(1) Let V be the vector space of continuous function on the interval $[0, \pi]$, that vanish at 0 and π; and let $\langle \, , \, \rangle$ be the scalar product defined by

$$\langle f, g \rangle = \int_0^{\pi} f(x) g(x) \, dx \, .$$

Let $g_k(x) = \sin(kx)$, for $k = 1, 2, 3, \ldots$.

(a) Show that $\{g_k : k = 1, 2, \ldots\}$ is an orthogonal set.

(b) Let $W_n \subset V$ be the subspace generated by the set $\{g_k : k = 1, 2, \ldots, n\}$, and let $f(x) = x(\pi - x)$. Let f_n denote the orthogonal projection of f onto W_n. Show that

$$f_{2n+1}(x) = \frac{8}{\pi} \sum_{k=0}^{n} \frac{\sin((2k+1)x)}{(2k+1)^3} \, .$$

(2) Let $\langle \, , \, \rangle$ be an inner product on a vector space V. Let $L : V \to V$ be a linear operator that satisfies the condition

$$\langle u, L(v) \rangle = \langle L(u), v \rangle \text{ for all } u, v, \in V \, .$$

Finally, let v_λ and v_μ be eigenvectors of L associated to two eigenvalues λ and μ, respectively. Prove that if $\lambda \neq \mu$, then $v_\lambda \perp v_\mu$.

(3) Let V be the vector space of continuous functions on the closed interval $[-1, 1]$, with scalar product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx \, .$$

(a) Apply the Gram-Schmidt orthogonalization process to the set $\{1, x, x^2, x^3\}$ to obtain an orthogonal set of four polynomials, $\{p_0(x), p_1(x), p_2(x), p_3(x)\}$.

(b) Verify that p_k is a solution of the differential equation

$$(1 - x^2) y'' - 2xy' + \lambda y = 0, \text{ with } \lambda = k(k+1) \, .$$

Remark: Applying Gram-Schmidt to the set $\{1, x, x^2, x^3, \ldots\}$ yields an orthogonal set $\{p_k(x) : k = 0, 1, 2, \ldots\}$ of polynomials, which after multiplication by constants are called *Legendre polynomials.* Moreover, $p_k(x)$ is a solution of the *Legendre equation*

$$(1 - x^2) y'' - 2xy' + \lambda y = 0 \text{ with } \lambda = k(k+1) \, .$$