Math 136: Homework 4

Due Thursday, April 21

(1) Let V be the vector space of 2×2 matrices, let $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, and define a linear map L by

$$\begin{aligned} L : V &\to V, \\ B &\mapsto BA. \end{aligned}$$

Find a basis for the kernel of L.

(2) Consider the linear map $L_A : \mathbf{R}^3 \to \mathbf{R}^3$, where $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 4 & 0 \\ -2 & -2 & 1 \end{pmatrix}$. Find the matrix associated to L_A with respect to the basis

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(3) The following exercises show that the set $\mathbf{C}$ of complex numbers can be represented as a set of 2×2 matrices of a certain form.

(a) Show that the complex numbers $\mathbf{C}$ under addition and multiplication by real numbers can be viewed as a 2-dimensional vector space.

(b) Let $M(2)$ denote the space of 2×2 matrices, and let $L : \mathbf{C} \to M(2)$ be the map defined by $L(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Verify that L is a linear map. (That is, $L(z_1 + z_2) = L(z_1) + L(z_2)$ and $L(cz) = cL(z)$, for all $z_1, z_2, z \in \mathbf{C}$ and $c \in \mathbf{R}$.

(c) Show that L satisfies the identity $L(z_1 z_2) = L(z_1)L(z_2)$ for all $z_1, z_2 \in \mathbf{C}$.

(d) What is the rank of L? What does this tell you about the kernel and image of L?

(4) Let $L : \mathbf{R}^3 \to M(3)$ be the linear map defined by $L((x, y, z)) = \begin{pmatrix} 0 & z & y \\ -z & 0 & x \\ -y & -x & 0 \end{pmatrix}$.

Observe that the image of L is the space $A(3)$ of 3×3, skew-symmetric matrices. Show that

$$L(\mathbf{u} \times \mathbf{v}) = [L(\mathbf{u}), L(\mathbf{v})],$$

where $[A, B]$ denotes the Lie bracket of A and B, i.e., $[A, B] = AB - BA$.