

Math 135: Homework 2

Due Thursday, January 13

- (1) Let  $\{a_n\}$  be the sequence defined inductively by  $a_1 = 1$ ,  $a_{n+1} = \frac{1}{a_n^4 + 16}$ .
- (a) Show that  $\{a_n\}$  is a Cauchy sequence.
  - (b) Show that  $\{a_n\}$  converges to a solution of the equation  $x^5 + 16x - 1 = 0$ .
  - (c) Show that if  $\{b_n\}$  is the sequence defined by  $b_1 = 2$ ,  $b_{n+1} = \frac{1}{b_n^4 + 16}$ , then  $\{b_n\}$  is convergent and  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n$ .

**Hint:** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 1/(x^4 + 16)$ .

- (2) Show that the equation

$$x = 1 + \int_0^x \frac{\cos(t)}{t^2 + 4} dt$$

has one and only one solution.

- (3) Recall that (by definition)  $\lim_{x \rightarrow \infty} f(x) = L$  if and only if for every real number  $\epsilon > 0$  there is a real number  $x_0$  such that  $|f(x) - L| < \epsilon$  for all  $x > x_0$ .

Prove the following:

$$\lim_{x \rightarrow \infty} f(x) = L \text{ if and only if } \lim_{t \rightarrow 0^+} f(1/t) = L.$$

- (4) Show that for any real number  $c$ ,

$$\lim_{x \rightarrow \infty} \left( \frac{x+c}{x-c} \right)^x = e^{2c}.$$