Final exam review

Topics:

- Sequences, series, improper integrals, and indeterminate forms. **Core material**: L'Hopital's rule; improper integrals; sequences: basic definitions and theorems about convergence, important limits; series: basic definitions, important series (geometric, harmonic, p-series), convergence tests (comparision, integral, root, ratio, alternating series); Taylor series and power series: radius of convergence, interval of convergence, important examples (like e^x, $\sin x$, $\cos x$, $\ln x$, geometric series), differentation and integration of series.
- Sequences, series, improper integrals, and indeterminate forms. **Other material**: fixed points of contractions, Abel's theorem (Theorem 12.9.5), binomial series.
- Differential equations. **Core material**: linear first order equations and their integrating factors; linear degree n equations: linear independence, form of the general solution in both the homogeneous and nonhomogeneous cases; solving linear constant coefficient homogeneous equations using the characteristic equation; solving linear constant coefficient nonhomogeneous equations using the method of undetermined coefficients; solving linear constant coefficient equations using Laplace transforms (using tables – the same one will be included), discontinuous forcing functions, δ-functions, convolutions; power series solutions, recursion formulas.
- Differential equations. **Other material**: variation of parameters; finding a second solution to a linear homogeneous equation using reduction of order. (No Picard iteration, no formulas from the sections on mass-spring systems.)
- Vector calculus. **Core material**: basic definitions (limit, derivative, integral); curves; unit tangent vector and principal normal vector; arc length; parametrization by arc length; curvature (definition on bottom of p. 725, formula 14.5.5 on p. 729; not 14.5.3 or 14.5.4). (No mechanics: no momentum, no angular momentum, no Kepler's laws.)

Two old Math 135 finals

(A1) The two curves $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ and $\mathbf{R}(s) = s^2\,\mathbf{i} + s^3\,\mathbf{j} + s^4\,\mathbf{k}$ intersect at the point $(1, 1, 1)$.

(a) Find parametric equations for each of the tangent lines to these curves at $(1, 1, 1)$.

(b) Find the angle between the two tangent lines to the curves at that point.

(c) Find the equation of the plane containing these two tangent lines.

(A2) Consider the initial value problem $y'' + y = \sin(\omega t)$, $y(0) = y'(0) = 0$.

(a) Find the solution for $\omega \neq 1$.

(b) Find the solution for $\omega = 1$.

(A3) Solve the initial value problem $y'' + y = \frac{1}{\cos(t)}$, $y(0) = y'(0) = 0$, $|t| < \pi/2$.

(A4) Consider the power series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^k \ln(2k+2)}$.

(a) Find its radius of convergence.

(b) Find its interval of convergence.

(A5) Solve the initial value problem $y'' - ty = 0$, $y(0) = y'(0) = 0$ using power series.

(A6) Express the solution of the initial value problem $y'' - y = \tan(t)$, $y(0) = y'(0) = 0$, as the convolution of two functions. (*Do not attempt to evaluate the convolution.*)

(A7) Evaluate $\displaystyle\lim_{x\to 0} \frac{x\sin(x) - \sin(x^2)}{\sin(2x^4)}$ in any way you wish.

(A8) Let $y = y(t)$ be a function that satisfies the identity $y(t) = \displaystyle\int_0^t \frac{s}{\cos(y(s))}\,ds$.

(a) Find an initial value problem that the function $y = y(t)$ satisfies.

(b) Solve the initial value problem to find $y(t)$.

(B1) Suppose that a curve C is specified parametrically by a vector function $\mathbf{r}(t)$. What is the formula for the curvature κ? For full credit, do *not* express your answer in terms of an arc length parametrization.

(B2) Find the radius of convergence for the series

$$\sum_{k=0}^{\infty} \frac{(2k)^k}{k!} x^{2k}.$$

(B3) Define a curve by

$$\mathbf{r}(t) = \cos 3t\,\mathbf{i} + t\,\mathbf{j} - \sin 3t\,\mathbf{k}.$$

Find its *binormal* $\mathbf{B}(t)$: this is defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, where $\mathbf{T} = \mathbf{T}(t)$ is the unit tangent vector and $\mathbf{N} = \mathbf{N}(t)$ is the principal normal vector.

(Check your work: what should the dot product $\mathbf{T} \cdot \mathbf{B}$ be? How about $\mathbf{N} \cdot \mathbf{B}$? How about $\mathbf{B} \cdot \mathbf{B}$?)

(B4) (a) Find the general solution to the equation

$$y'''' - y = 0.$$

(b) Find the general solution to the equation

$$y'' - 3y' - 10y = 14e^{-2t}.$$

(B4) The initial value problem

$$(x+1)^2 y'' = x + 3, \quad y(0) = 0,\ y'(0) = 0$$

has a solution of the form $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Find a_n for $n \le 3$ and find the general recursion relation, making sure to specify the values of n for which it is valid.

(B5) (a) Solve the initial value problem

$$y'' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0,$$

where

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \cos 2t & \text{if } 0 \le t < 2\pi, \\ 0 & \text{if } t \ge 2\pi. \end{cases}$$

(b) Sketch the solution.