1 Preliminaries

If $f(t)$ is defined on the interval $[0, \infty)$, then its Laplace transform is defined to be

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt,$$

as long as this integral is defined and converges. In particular, if f is of *exponential order* and is *piecewise continuous*, the Laplace transform of $f(t)$ will be defined.

- f is of *exponential order* if there are constants M and c so that

  $$|f(t)| \le Me^{ct}.$$

  Since the integral $\int_0^\infty e^{-st} Me^{ct}\, dt$ converges if $s > c$, then by a comparison test (like (11.7.2) in Salas-Hille-Etgen), the integral defining the Laplace transform of $f(t)$ will converge.

- f is *piecewise continuous* if over each interval $[0, b]$, $f(t)$ has only finitely many discontinuities, and at each point a in $[0, b]$, both of the limits

  $$\lim_{t \to a^-} f(t) \quad \text{and} \quad \lim_{t \to a^+} f(t)$$

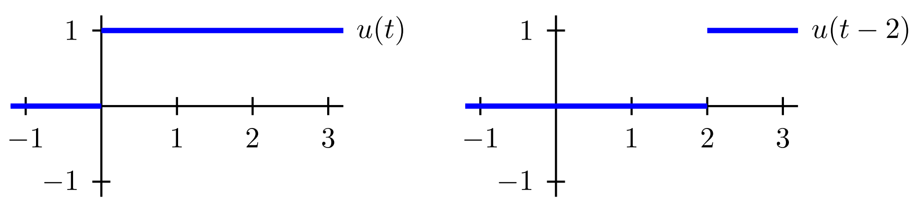
  exist – they need not be equal, but they must exist. (At the endpoints 0 and b, the appropriate one-sided limits must exist.)

2 Step functions

Define $u(t)$ to be the function

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \ge 0. \end{cases}$$

Then $u(t)$ is called the *step function*, or sometimes the *Heaviside step function*: it jumps from 0 to 1 at $t = 0$. Note that for any number $a > 0$, the graph of the function $u(t - a)$ is the same as the graph of $u(t)$, but translated right by a: $u(t - a)$ jumps from 0 to 1 at $t = a$.



Example 1. We can use the step function to write formulas for functions which are defined piecewise: if $g(t)$ is defined as

$$g(t) = \begin{cases} L(t) & \text{if } t < a, \\ R(t) & \text{if } t \ge a, \end{cases}$$

then

$$g(t) = L(t) + [R(t) - L(t)]\, u(t-a).$$

Why? If $t < a$, then the step function $u(t-a)$ is zero, so this turns into $L(t)$. If $t \geq a$, then the step function equals 1, so the $L(t)$ terms cancel and we are left with $R(t)$. We can get similar expressions for functions defined in three or more separate pieces.

Example 2. Express the function

$$g(t) = \begin{cases} t & \text{if } t < 1, \\ t^2 & \text{if } 1 \leq t < 3, \\ t^3 & \text{if } t \geq 3 \end{cases}$$

using the step function.

We will work from left to right on the real line. When $t < 1$, the expression t is correct, but it's not right for $t \geq 1$. Between 1 and 3, we have to subtract t and add t^2. We can do this by adding an appropriate function multiplied by $u(t-1)$, and in particular, this expression is correct when $t < 3$:

$$t + u(t-1)\left[t^2 - t\right].$$

Finally, when $t \geq 3$, we need to subtract t^2 and add t^3, so we get this formula for $g(t)$:

$$g(t) = t + u(t-1)\left[t^2 - t\right] + u(t-3)\left[t^3 - t^2\right].$$

Proposition 3. *The Laplace transform of $u(t-a)$ is e^{-as}/s. If $f(t)$ is a function with Laplace transform $F(s)$, then*

$$\mathcal{L}\left(u(t-a)f(t-a)\right) = e^{-as}F(s).$$

Proof. The integral defining the Laplace transform is

$$\mathcal{L}\left(u(t-a)f(t-a)\right) = \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt = \int_a^\infty e^{-st}f(t-a)\,dt.$$

(The second equality is by the definition of the step function.) Now make a change of variables: let $w = t - a$. When $t = a$, $w = 0$, and when $t = \infty$, $w = \infty$, so the integral becomes

$$\int_0^\infty e^{-s(w+a)}f(w)\,dw = \int_0^\infty e^{-sw}e^{-sa}f(w)\,dw = e^{-sa}\int_0^\infty e^{-sw}f(w)\,dw = e^{-sa}\mathcal{L}\left(f(t)\right).$$

□

Example 4. $\mathcal{L}\left(u(t-3)(t-3)^2\right) = e^{-3s}\dfrac{2!}{s^3}$.

By the way, since the Laplace transform is defined in terms of an integral, the behavior at the discontinuities of piecewise-defined functions is not important. For example, the following functions will have the same Laplace transform:

$$g(t) = \begin{cases} 0 & \text{if } t < 1, \\ t & \text{if } t \geq 1, \end{cases} \qquad h(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ t & \text{if } t > 1. \end{cases}$$

Exercises 5. (a) Suppose $f(t)$ is some function. In terms of the graph of $f(t)$, what does the function $u(t-a)f(t-a)$ look like?

(b) Compute $\mathcal{L}(u(t-2))$.

(c) Compute $\mathcal{L}(u(t-2)\sin(t-2))$.

Example 6. What is the Laplace transform of the function $g(t)$ defined by

$$g(t) = \begin{cases} 0 & \text{when } t < 1, \\ t-1 & \text{when } t \geq 1? \end{cases}$$

To answer this, we need to write $g(t)$ in terms of the step function, but that's easy here: $g(t) = u(t-1)(t-1)$. So in the notation of Proposition 3, we could write this as $g(t) = u(t-1)f(t-1)$, where $f(t) = t$. So by the proposition,

$$\mathcal{L}(g(t)) = e^{-s}\mathcal{L}(t) = e^{-s}\frac{1}{s^2}.$$

Example 7. Suppose that $g(t)$ is the function defined by

$$g(t) = \begin{cases} t & \text{if } t < 1, \\ t^2 & \text{if } 1 \leq t < 3, \\ t^3 & \text{if } t \geq 3 \end{cases}$$

What is $\mathcal{L}(g(t))$?

In Example 2 we found an expression for $g(t)$ in terms of step functions:

$$g(t) = t + u(t-1)(t^2 - t) + u(t-3)(t^3 - t^2).$$

Unfortunately, this isn't in the right form to apply the proposition, because the terms don't look like $u(t-a)f(t-a)$ for some function f. Let's look at the summands: the first summand is just t, and we can compute its Laplace transform: it's just $1/s^2$. The second summand is $u(t-1)(t^2-t)$, and so we need to write $t^2 - t$ as a function of $t-1$: we want a function f so that $f(t-1) = t^2 - t$. If we make the substitution $x = t-1$, then $t = x+1$, and this becomes $f(x) = (x+1)^2 - (x+1) = x^2 + x$. This is our formula for f: $f(x) = x^2 + x$, and so $f(t) = t^2 + t$.

Similarly, the third summand is $u(t-3)(t^3 - t^2)$, and so we want to write $t^3 - t^2$ as $f_2(t-3)$ for some function f_2. We want $f_2(t-3) = t^3 - t^2$, so letting $x = t-3$, we get $f_2(x) = (x+3)^3 - (x+3)^2 = x^3 + 8x^2 + 21x + 18$.

Now we can compute the Laplace transform of $g(t)$:

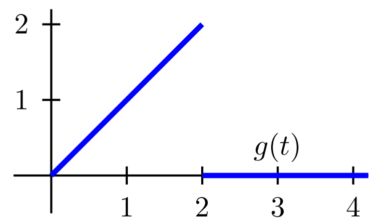
$$\begin{aligned} \mathcal{L}(g(t)) &= \mathcal{L}(t + u(t-1)f(t-1) + u(t-3)f_2(t-3)) \\ &= \mathcal{L}(t) + \mathcal{L}(u(t-1)f(t-1)) + \mathcal{L}(u(t-3)f_2(t-3)) \\ &= \frac{1}{s^2} + e^{-s}\mathcal{L}(f(t)) + e^{-3s}\mathcal{L}(f_2(t)) \\ &= \frac{1}{s^2} + e^{-s}\mathcal{L}\left(t^2 + t\right) + e^{-3s}\mathcal{L}\left(t^3 + 8t^2 + 21t + 18\right) \\ &= \frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) + e^{-3s}\left(\frac{3!}{s^4} + \frac{8 \cdot 2!}{s^3} + \frac{21}{s^2} + \frac{18}{s}\right). \end{aligned}$$

Example 8. Suppose that $g(t)$ is defined by

$$g(t) = \begin{cases} t & \text{when } 0 \le t < 2, \\ 0 & \text{when } t \ge 2. \end{cases}$$

Solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0.$$



I would like to use Laplace transforms, and to compute the Laplace transform of $g(t)$, I need to write $g(t)$ using step functions. When computing the Laplace transform of a function $g(t)$, what happens when $t < 0$ is irrelevant – the integral starts at $t = 0$ – so I can write $g(t)$ like this:

$$g(t) = t + u(t-2)(-t) = t + u(t-2)(-(t-2) - 2).$$

So if $f(t) = -t - 2$, then $g(t) = t + u(t-2)f(t-2)$. Now we're ready to go: let $Y = \mathcal{L}(y)$, and then because of the initial conditions, $\mathcal{L}(y') = sY$ and $\mathcal{L}(y'') = s^2Y$. When I apply the Laplace transform to the differential equation, I get

$$(s^2+4)Y = \mathcal{L}(t + u(t-2)f(t-2)) = \frac{1}{s^2} + e^{-2s}\left(-\frac{1}{s^2} - \frac{2}{s}\right).$$

Therefore

$$\begin{aligned} Y &= (1 - e^{-2s})\frac{1}{s^2(s^2+4)} - 2e^{-2s}\frac{1}{s(s^2+4)} \\ &= (1 - e^{-2s})\left(\frac{1/4}{s^2} + \frac{-1/4}{s^2+4}\right) - 2e^{-2s}\left(\frac{1/4}{s} + \frac{-s/4}{s^2+4}\right) \\ &= \left(\frac{1/4}{s^2} + \frac{-1/4}{s^2+4}\right) + e^{-2s}\left(-\frac{1/4}{s^2} + \frac{1/4}{s^2+4} - \frac{1/2}{s} + \frac{s/2}{s^2+4}\right). \end{aligned}$$

Therefore

$$\begin{aligned} y &= \mathcal{L}^{-1}\left(\frac{1/4}{s^2} + \frac{-1/4}{s^2+4}\right) + \mathcal{L}^{-1}\left(e^{-2s}\left(-\frac{1/4}{s^2} + \frac{1/4}{s^2+4} - \frac{1/2}{s} + \frac{s/2}{s^2+4}\right)\right) \\ &= \frac{1}{4}t - \frac{1}{8}\sin 2t + u(t-2)f(t-2), \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1}\left(-\frac{1/4}{s^2} + \frac{1/4}{s^2+4} - \frac{1/2}{s} + \frac{s/2}{s^2+4}\right)$$
$$= -\frac{1}{4}t + \frac{1}{8}\sin 2t - \frac{1}{2} + \frac{1}{2}\cos 2t.$$

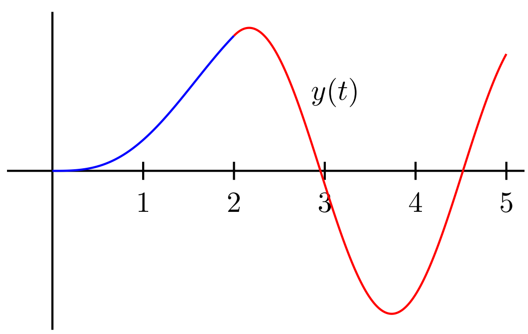
So

$$y(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t + u(t-2)\left(-\frac{1}{4}(t-2) + \frac{1}{8}\sin 2(t-2) - \frac{1}{2} + \frac{1}{2}\cos 2(t-2)\right)$$
$$= \frac{1}{4}t - \frac{1}{8}\sin 2t + u(t-2)\left(-\frac{1}{4}t + \frac{1}{8}\sin 2(t-2) + \frac{1}{2}\cos 2(t-2)\right).$$

We can also write this as a piecewise-defined function:

$$y(t) = \begin{cases} \frac{1}{4}t - \frac{1}{8}\sin 2t & \text{if } t < 2, \\ -\frac{1}{8}\sin 2t + \frac{1}{8}\sin 2(t-2) + \frac{1}{2}\cos 2(t-2) & \text{if } t \geq 2. \end{cases}$$

You can check that this function is continuous and twice differentiable for all $t \geq 0$. (The only interesting point is $t = 2$.) Here's a graph:



Example 9. Suppose that $g(t)$ is defined by

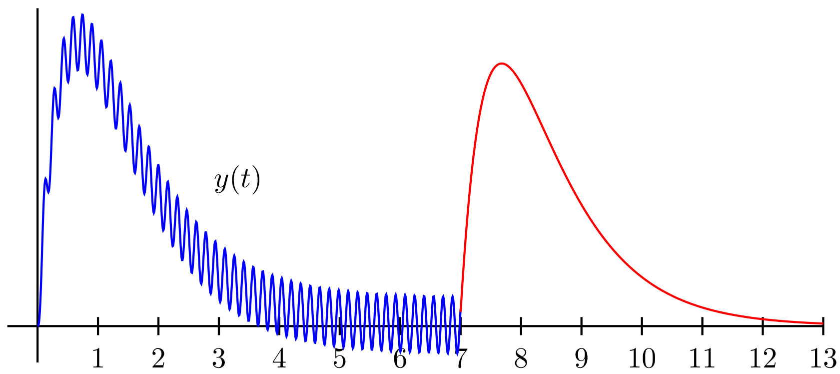
$$g(t) = \begin{cases} 100\sin 40t & \text{when } 0 \leq t < 7, \\ 0 & \text{when } t \geq 7. \end{cases}$$

Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \quad y(0) = 0,\ y'(0) = 0.$$

I'm going to skip the details and just reproduce the solution and its graph; we'll discuss some interpretations of this example in class:

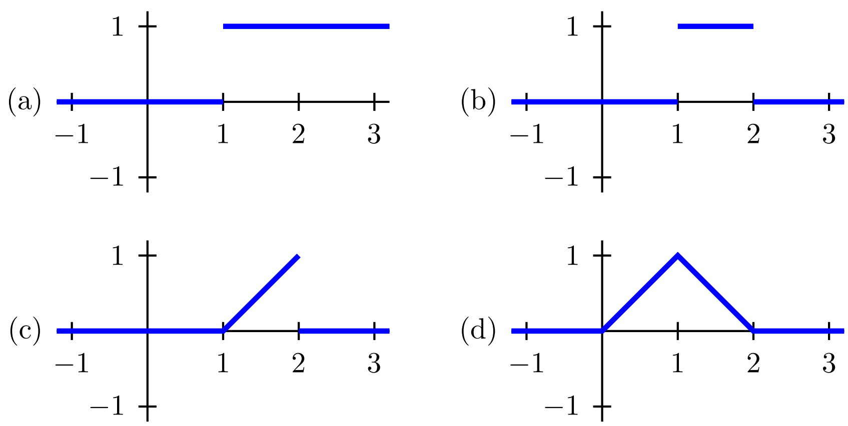
$$y = \begin{cases} -2.49e^{-2t} + 2.50e^{-t} - 0.0622\sin(40t) - 0.00467\cos(40t), & \text{if } 0 \leq t < 7, \\ -2.25e^{-2(t-7)} + 2.28e^{-(t-7)}, & \text{if } t \geq 7. \end{cases}$$



Exercises 10. Solve the initial value problems

$$y'' + y = f(t), \quad y(0) = 0, \; y'(0) = 0,$$

where $f(t)$ is as given in each of the following pictures.



3 The Dirac δ-function

Proposition 11. *(a) Let ϵ be a positive number and consider the function $f_\epsilon(t)$ defined by*

$$f_\epsilon(t) = \begin{cases} 1/\epsilon & \text{if } 0 \le t \le \epsilon, \\ 0 & \text{if } t > \epsilon. \end{cases}$$

Then

$$\mathcal{L}\left(f_\epsilon(t)\right) = \frac{1 - e^{s\epsilon}}{s\epsilon}.$$

(b) "Define" the Dirac delta function $\delta(t)$ *to be*

$$\delta(t) = \lim_{\epsilon \to 0^+} f_\epsilon(t).$$

Then $\delta(t) = 0$ except when $t = 0$, and it has the following properties with respect to integration: for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \quad \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).$$

Therefore for any positive number a, we have $\mathcal{L}\left(\delta(t-a)\right) = e^{-as}$.

The Dirac delta function models an instantaneous force applied to a system, like hitting a mass with a hammer.

Now, $\delta(t)$ is not actually a function: the limit defining it doesn't exist when $t = 0$, for one thing. If there were a way to define it, then properties of integrals show that if $g(t)$ is any function with $g(t) = 0$ whenever $t \neq 0$, then $\int_a^b g(t) = 0$ for any a and b. Instead, $\delta(t)$ is what is called a *generalized function* or *distribution*, and although it isn't a function, it can be treated like one in many ways. Really its defining property is that for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).$$

Example 12. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-1), \quad y(0) = 0,\ y'(0) = 0.$$

We apply the Laplace transform. If $Y = \mathcal{L}(y)$, then $\mathcal{L}(y') = sY$ and $\mathcal{L}(y'') = s^2 Y$, and the equation becomes

$$(s^2 + 2s + 2)Y = e^{-s}, \quad \text{so} \quad Y = e^{-s}\frac{1}{s^2 + 2s + 2}.$$

We complete the square and write this as

$$Y = e^{-s}\frac{1}{s^2 + 2s + 2} = e^{-s}\frac{1}{(s+1)^2 + 1}.$$

Therefore

$$y = \mathcal{L}^{-1}(Y) = u(t-1) f(t-1),$$

where

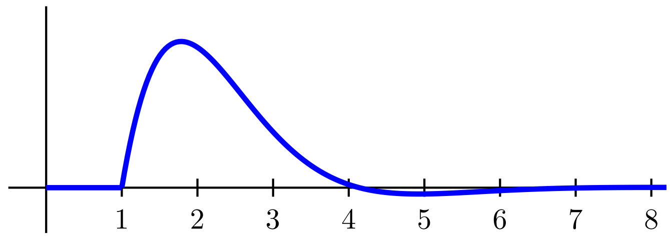
$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1}\right) = e^{-t}\sin t.$$

So

$$y(t) = u(t-1)e^{-(t-1)}\sin(t-1),$$

or

$$y(t) = \begin{cases} 0 & \text{if } t < 1, \\ e^{-(t-1)}\sin(t-1) & \text{if } t \geq 1. \end{cases}$$



Note that this function is continuous everywhere, but it is not differentiable at $t = 1$. This is not surprising, because $t = 1$ is when the delta function is applied – this example models what happens in a damped spring system when you hit the mass with a hammer.

Exercises 13. (a) Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-1), \quad y(0) = 0,\ y'(0) = 0.$$

(b) Consider the initial value problem

$$y'' + 4y = \delta(t) + c\delta(t-\pi), \quad y(0) = 0,\ y'(0) = 0,$$

where c is a constant. What should c be so that the solution stops completely at time π? That is, what should c be so that the solution has the form

$$y = \begin{cases} ??? & \text{if } 0 \le t < \pi, \\ 0 & \text{if } t \ge \pi? \end{cases}$$