The complex numbers $\mathbb{C}$ are important in just about every branch of mathematics. These notes[1] present some basic facts about them.

1 The Complex Plane

A *complex number* z is given by a pair of real numbers x and y and is written in the form $z = x + iy$, where i satisfies $i^2 = -1$. The complex numbers may be represented as points in the plane, with the real number 1 represented by the point $(1, 0)$, and the complex number i represented by the point $(0, 1)$. The x-axis is called the "real axis," and the y-axis is called the "imaginary axis." For example, the complex numbers 1, i, $3 + 4i$ and $3 - 4i$ are illustrated in Fig 1a.

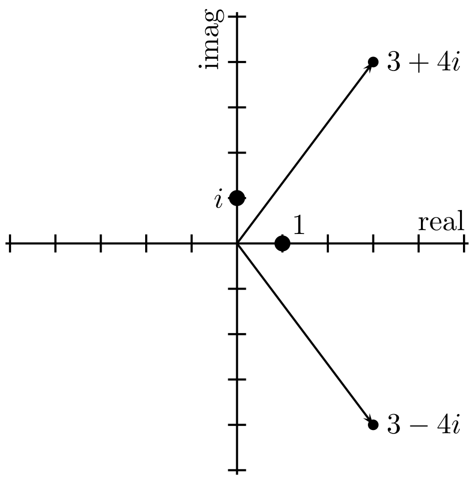


Fig 1a

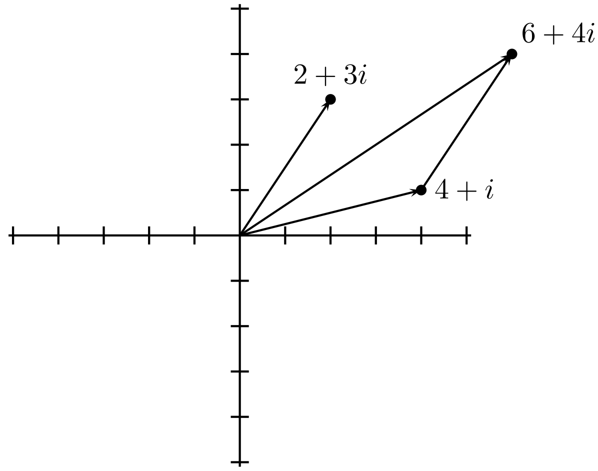


Fig 1b

Complex numbers are added in a natural way: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{1}$$

It's just vector addition. Fig 1b illustrates the addition $(4+i)+(2+3i) = (6+4i)$. Multiplication is given by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2 + i)(-2 + 4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i$$

We call x the *real part* of z and y the *imaginary part*, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. (**Remember**: $\operatorname{Im} z$ is a *real* number.) The term "imaginary" is a historical holdover; it took mathematicians some time to accept the fact that i (for "imaginary," naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

[1] Based on notes written by Bob Phelps, with modifications by Tom Duchamp and John Palmieri.

There is only one way we can have $z_1 = z_2$, namely, if $x_1 = x_2$ and $y_1 = y_2$. An equivalent statement is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If a is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what we would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$). Division is more complicated (although we will show later that the *polar representation* of complex numbers makes it easy). To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1. The rule for finding the reciprocal of $z = x + iy$ is given by:

$$\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} \tag{2}$$

The expression $x - iy$ appears so often and is so useful that it is given a name: it is called the *complex conjugate* of $z = x + iy$, and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3+4i} = 3 - 4i$, as illustrated in FIG 1A . Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.

Another important quantity associated with a complex number z is its *modulus* (also known as its *absolute value* or *magnitude*):

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2+y^2} = \big((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\big)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. This leads to the inequality

$$\operatorname{Re} z \le |\operatorname{Re} z| = \sqrt{(\operatorname{Re} z)^2} \le \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z| \tag{3}$$

Similarly, $\operatorname{Im} z \le |\operatorname{Im} z| \le |z|$.

Exercises 1.

1. Show that the product of $z = x + iy$ and the expression (2) above equals 1.
2. Verify each of the following:
   (a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$ (b) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$
   (c) $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{1}{2}i$ (d) $(1-i)^4 = -4$
3. Prove the following:
   (a) $z + \overline{z} = 2\operatorname{Re} z$, and z is a real number if and only if $\overline{z} = z$. (Note also that $z - \overline{z} = 2i\operatorname{Im} z$.)
   (b) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.
4. Prove that $|z_1 z_2| = |z_1||z_2|$ (Hint: Use (3b).)
5. Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

2 Polar Representation of Complex Numbers

Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) is

$$
\begin{aligned}
x &= r\cos\theta \qquad \text{and} \qquad y = r\sin\theta, \\
r &= \sqrt{x^2+y^2} \qquad \text{and} \qquad \theta = \arctan\frac{y}{x}.
\end{aligned}
$$

(If $(x, y) = (0, 0)$, then $r = 0$ and θ can be anything.) This means that for the complex number $z = x + iy$, we can write

$$z = r(\cos\theta + i\sin\theta).$$

There is another way to rewrite this expression for z. We know that for any real number x, e^x can be expressed as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

For any complex number z, we *define* e^z by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots .$$

In particular,

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}
$$

This is Euler's Formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}.$$

For example,

$$e^{i\pi/2} = i, \qquad e^{\pi i} = -1 \qquad \text{and} \qquad e^{2\pi i} = 1.$$

Given $z = x + iy$, then z can be written in the form $z = re^{i\theta}$, where

$$r = \sqrt{x^2+y^2} = |z| \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}. \tag{4}$$

That is, r is the magnitude of z. **Note**: In the polar representation of complex number, we *always* assume that r is non-negative. The angle θ is sometimes called the *argument* or *phase* of z.

For example, the complex number $z = 8 + 6i$ may also be written as $10e^{i\theta}$, where $\theta = \arctan(.75) \approx 0.64$ radians, as illustrated in Fig 2.

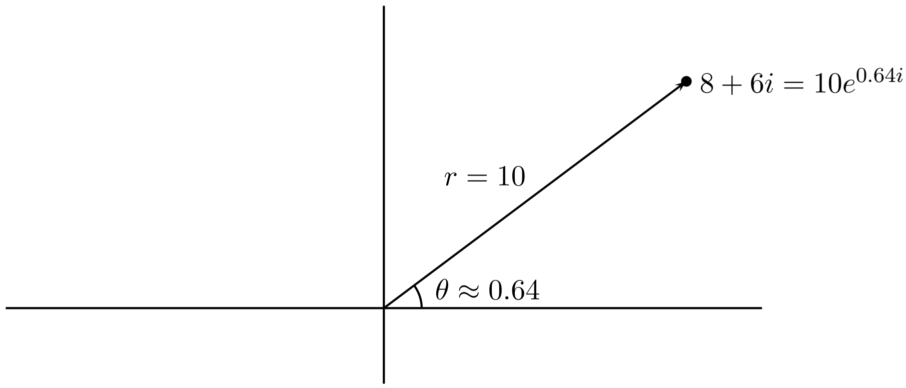


Fig 2

If $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$; therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle which differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus $-4 + 4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$. In general, if $z = re^{i\theta}$, then we also have $z = re^{i(\theta+2\pi k)}$, $k = 0, \pm 1, \pm 2, \ldots$. Moreover, there is ambiguity in equation (4) about the inverse tangent which can (and *must*) be resolved by looking at the signs of x and y, respectively, in order to determine the quadrant in which θ lies. If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$. Despite this, the polar representation is very useful when it comes to multiplication:

$$\text{if } z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \quad \text{then} \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} \tag{5}$$

That is, to obtain the product of two complex numbers, *multiply their moduli and add their angles*. To see why this is true, write $z_1z_2 = re^{i\theta}$, so that $r = |z_1z_2| = |z_1||z_2| = r_1r_2$ (by Exercise (4a)). It remains to show that $\theta = \theta_1 + \theta_2$, that is, that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ (this is Exercise (7a) below). For example, let

$$z_1 = 2 + i = \sqrt{5}e^{i\theta_1}, \quad \theta_1 \approx 0.464, \qquad z_2 = -2 + 4i = \sqrt{20}e^{i\theta_2}, \quad \theta_2 \approx 2.034.$$

If $z_3 = z_1z_2$, then $r_3 = r_1r_2$ and $\theta_3 = \theta_1 + \theta_2$; that is,

$$z_3 = -8 + 6i = \sqrt{100}e^{i\theta_3}, \quad \theta_3 \approx 2.498,$$

as shown in the picture.

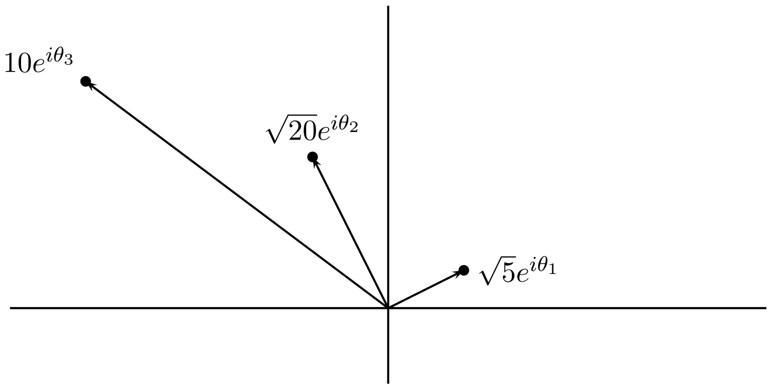


FIG 3

Applying (5) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2}e^{\frac{3}{4}\pi i}$ (our earlier example), we get

$$(4 + 4i)^2 = (4\sqrt{2}e^{\frac{3}{4}\pi i})^2 = 32e^{\frac{3}{2}\pi i} = -32i.$$

By an easy induction argument, the formula in (5) can be used to prove that for any positive integer n,

$$\text{If } z = re^{i\theta}, \text{ then } z^n = r^n e^{in\theta}.$$

This makes it easy to solve equations like $z^4 = -7$. Indeed, writing the unknown number z as $re^{i\theta}$, we have $r^4e^{i4\theta} = -7 \equiv 7e^{\pi i}$, hence $r^4 = 7$ (so $r = 7^{1/4}$, since r must be a non-negative real number) and $4\theta = \pi + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$. It follows that $\theta = \pi/4 + 2k\pi/4$, $k = 0, \pm1, \pm2, \ldots$. There are only four distinct complex numbers of the form $e^{(\pi/4+k\pi/2)i}$, namely $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$ and $e^{7\pi i/4}$. The first of the following figures illustrates $z = -7$ and its four fourth roots $z_1 = 7^{1/4}e^{\pi i/4}$, $7^{1/4}e^{3\pi i/4}$, $7^{1/4}e^{5\pi i/4}$ and $7^{1/4}e^{7\pi i/4}$, all of which lie on the circle of radius $7^{1/4}$ about the origin.

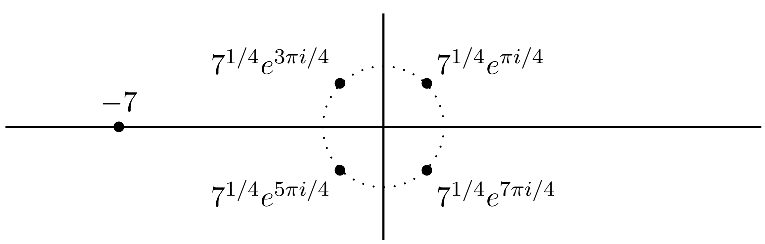


FIG 4

From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

By expanding on the left and equating real and imaginary parts, you obtain trigonometric identities which can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ and looking at the real part produces $\cos 2\theta = \cos^2\theta - \sin^2\theta$. For $n = 3$ one gets $\cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos^2\theta$.

Let's also note the following formulas: if $z = re^{i\theta}$, then

$$\overline{z} = re^{-i\theta}, \quad \operatorname{Re} z = r\cos\theta, \quad \operatorname{Im} z = r\sin\theta.$$

Combined with the formulas from Exercise (3a), we get

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = -\frac{i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

Exercises 2.

(6) Let $z_1 = 3i$ and $z_2 = 2 - 2i$

  (a) Plot the points $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.
  (b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
  (c) Express z_1 and z_2 in polar form.

(7) Prove the following:

  (a) $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.
  (b) Use (a) to show that $(e^{i\theta})^{-1} = e^{-i\theta}$, that is, $e^{-i\theta}e^{i\theta} = 1$.

(8) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, z_1z_2 and z_1/z_2.

(9) Find all complex numbers z which satisfy $z^3 = -1$.

(10) Find all complex numbers z such that $z^2 = \sqrt{2}e^{i\pi/4}$.

3 Complex-valued Functions

Now suppose that $w = w(t)$ is a complex-valued function of the real variable t. That is,

$$w(t) = u(t) + iv(t)$$

where $u(t)$ and $v(t)$ are real-valued functions. A complex-valued function defines a curve in the complex plane.

The derivative of $w(t)$ with respect to t is *defined* to be the function

$$w'(t) = u'(t) + iv'(t)$$

(This is just like the definition of the derivative of a vector-valued function – just differentiate the components.) The derivative can be viewed as the tangent vector to the complex curve.

It is easily checked (just expand the left and right hand sides of each identity) that the following formulas hold for complex-valued functions $z = z(t)$ and $w = w(t)$:

$$C' = 0 \text{ where } C = \text{constant}$$
$$(z + w)' = z' + w'$$
$$(zw)' = z'w + zw'$$
$$(Cz)' = Cz' \text{ where } C = \text{constant}$$
$$(z^n)' = nz^{n-1}z'$$

One function is of particular interest to us: the *complex exponential function*. It is defined as follows:

$$e^{(\rho+i\omega)t} = e^{\rho t}e^{i\omega t} = e^{\rho t}\cos(\omega t) + ie^{\rho t}\sin(\omega t)\,.$$

The corresponding curve in the complex plane is a spiral curve: the quantity ω is the angular velocity of the spiral ($\omega > 0$ corresponds to a counterclockwise spiral, $\omega < 0$ to a clockwise one). The quantity ρ measures the rate at which the spiral expands outward ($\rho > 0$) or inward ($\rho < 0$).

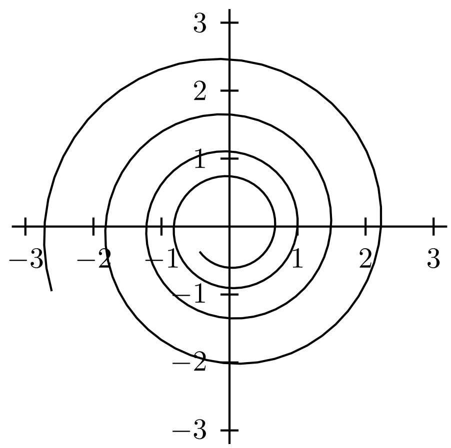


FIG 5

Computing the derivatives of the real and imaginary parts and collecting terms yields the formula $\left(e^{(\rho+i\omega)t}\right)' = (\rho+i\omega)e^{(\rho+i\omega)t}$. In other words, *even if r is a complex number, the formula*

$$\frac{d}{dt}e^{rt} = re^{rt}$$

holds!

4 The function $x(t) = e^{\rho t}(C_1\cos(\omega t) + C_2\sin(\omega t))$

We want to write the function

$$x(t) = C_1e^{\rho t}\cos(\omega t) + C_2e^{\rho t}\sin(\omega t)$$

in the form

$$x(t) = Ae^{\rho t}\cos(\omega t - \phi),$$

because then we know what the graph of $x(t)$ looks like.

First notice that

$$Ae^{\rho t}\cos(\omega t - \phi) = (A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t))\; e^{\rho t},$$

so let

$$A\cos(\phi) = C_1 \text{ and } A\sin(\phi) = C_2.$$

Then we get

$$A = \sqrt{C_1^2 + C_2^2} \text{ and } \tan(\phi) = \frac{C_2}{C_1}\,.$$

Example 1. Consider the function

$$x(t) = (5\cos(2t) + 4\sin(2t))e^{-t/5}\,.$$

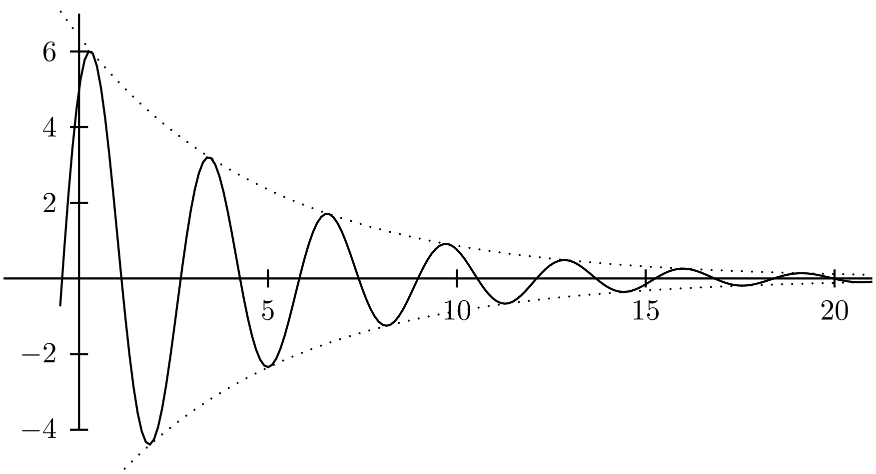
The point $(C_1, C_2) = (5, 4)$ is in the first quadrant so $0 < \phi < \pi/2$. So

$$A = \sqrt{5^2 + 4^2} = \sqrt{41} \text{ and } \phi = \tan^{-1}(4/5)\,.$$

Hence,

$$(5\cos(2t) + 4\sin(2t))e^{-t/5} = \sqrt{41}\,e^{-t/5}\cos\left(2t - \tan^{-1}(4/5)\right)\,.$$

Here is a sketch of this curve, showing it oscillating between $\sqrt{41}e^{-t/5}$ and $-\sqrt{41}e^{-t/5}$:



Note: There is an alternate description of $x(t)$ that makes direct use of the polar form of complex numbers. Since $5\cos(2t)e^{-t/5}$ is the real part of $5e^{(-1/5+2i)t}$ and since $4\sin(2t)e^{-t/5}$ is the real part of $-4ie^{(-1/5+2i)t}$, let $C = 5 - 4i$ and $\rho + i\omega = -1/5 + 2i$. Then

$$x(t) = \operatorname{Re}\left((5 - 4i)e^{(-1/5+2i)t}\right)$$

Of course the earlier expression, $\sqrt{41}\,e^{-t/5}\cos\left(2t - \tan^{-1}(4/5)\right)$, is easier to graph.

Exercises 3.

(11) Sketch the graph of the curve

$$z(t) = (2 + 2i)e^{\left(\frac{1}{2} + \pi i\right)t}$$

for $0 \leq t \leq 3$. Sketch the graph of $x = x(t) = \operatorname{Re}(z(t))$.

(12) Consider the function

$$x(t) = 3e^{-2t}\cos(4t) - 5e^{-2t}\sin(4t)\,.$$

Write it in each of the forms

$$x(t) = Ae^{\rho\, t} \cos(\omega\, t - \phi)$$

and

$$x(t) = \mathrm{Re}\left(Ce^{rt}\right)$$

where A, ω and ϕ are real numbers and C and r are complex numbers.