The point of this handout is Theorem 2: a way of proving that a sequence converges even if you can't tell what the limit is.

Definition 1. A sequence $\{a_n\}$ is said to be *Cauchy* (or to be a *Cauchy sequence*) if for every real number $\epsilon > 0$, there is an integer N (possibly depending on ϵ) for which

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N\,. \tag{1}$$

Theorem 2. *A sequence of real numbers is convergent if and only if it is Cauchy.*

Proof. ($\Rightarrow$) Let $\{a_n\}$ be a convergent sequence with limit L. To verify that $\{a_n\}$ is Cauchy, begin by choosing a number $\epsilon > 0$. We must show that there is an integer N for which (1) holds.

But since a_n converges to L, there is an integer $N > 0$ for which $|a_n - L| < \epsilon/2$ for all $n \geq N$. Notice that for all $n, m > N$ we may estimate as follows:

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) - (a_m - L)| \\ &\leq |a_n - L| + |a_m - L| \quad \text{(by the triangle inequality)} \\ &< \epsilon/2 + \epsilon/2 = \epsilon\,. \end{aligned}$$

Thus $\{a_n\}$ is Cauchy.

($\Leftarrow$) Now let $\{a_n\}$ be a Cauchy sequence. We want to show that $\{a_n\}$ converges.

First notice that $\{a_n\}$ is bounded. To see this, let $\epsilon = 1$. Then there is an integer N such that $|a_n - a_m| < 1$ for all $n, m > N$. Set $m = N + 1$; then for all $n > N$, we have

$$(a_{N+1}) - 1 < a_n < (a_{N+1}) + 1.$$

Let $U = \max\{a_1, \ldots, a_N, a_{N+1} + 1\}$ and $L = \min\{a_1, \ldots, a_N, a_{N+1} - 1\}$. Clearly, $L \leq a_n \leq U$ for all n, so $\{a_n\}$ is bounded.

Now let $\{b_n\}$ and $\{c_n\}$ be the bounded sequences defined by

$$b_n = \operatorname*{glb}_{m \geq n} a_m \quad \text{and} \quad c_n = \operatorname*{lub}_{m \geq n} a_m\,,$$

and notice that by construction the following inequalities are satisfied:

$$b_n \leq a_m \leq c_n \text{ for all } m \geq n. \tag{2}$$

Finally notice that $\{b_n\}$ is nondecreasing and $\{c_n\}$ is nonincreasing (see Exercise 29, page 532). Hence, by Theorem 11.3.6, both $\{b_n\}$ and $\{c_n\}$ converge. Let $B = \lim_{n\to\infty} b_n$ and $C = \lim_{n\to\infty} c_n$. By the theorem, the following inequalities hold:

$$b_n \leq B \leq C \leq c_n \text{ for all } n.$$

We will show that $\{a_n\}$ converges by showing $B = C$ and applying the pinching lemma.

Begin by choosing any $\epsilon > 0$. Then there is an integer N for which

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N.$$

In particular,

$$a_N - \epsilon < a_m \text{ for all } m \geq N,$$

showing that $a_N - \epsilon \leq b_N \leq B$. Similarly

$$a_m < a_N + \epsilon \text{ for all } m \geq N,$$

showing that $C \leq c_N \leq a_N + \epsilon$. Hence,

$$a_N - \epsilon < B \leq C < a_N + \epsilon\,.$$

It follows that $C - B < 2\epsilon$ for every $\epsilon > 0$, which implies that $B = C$.

Since $b_n \leq a_n \leq c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = B$, the pinching lemma applies to show that $a_n \to B$, concluding the proof. □