Math 327: Real numbers and limits

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Introduction

The textbook for this course is *Advanced Calculus* by Taylor & Mann. That book doesn't work perfectly – I'll explain the problems – and these notes are intended to patch the flaws.

The first issue with the book is that it assumes that you've seen limits before, not just casually as in Math 124, but done carefully with a precise mathematical definition. So while the authors include limits in Section 1.62 of Chapter 1, their "review" chapter, it makes more sense for us to introduce them after we've established some basic properties of the real numbers.

The second issue with the book is really a difference of opinions: I prefer a slightly different viewpoint on the real numbers than is given in Chapter 2 of the book.

So these notes are a replacement for Chapter 2 and Section 1.62 of the book. Actually, the notes aren't as long as these parts of the book, so you should probably read the two of them together – I've tried to give specific suggestions at various points in the notes. This is especially true when we get to the material on limits: the book has more examples than these notes do.

The starting point: we assume that we know about the set **Z** of integers and the set **Q** of rational numbers, and we assume that we've heard of the set **R** of real numbers. The set **C** of complex numbers appears every now and then as an example, but it won't be used very heavily. We could instead start almost from scratch: starting with the basic rules for sets, it is possible to construct the non-negative integers, and from them all integers. Once we have all integers, we could construct the rational numbers, and once we have the rationals, we could construct the reals. This last step in particular is rather involved, and the whole process would take us a bit far afield, and we are not going to do it. We skip the proofs of several major theorems below, because to prove them we would really need this sort of precise construction of the real numbers. Ask me if you are interested in more information about these constructions (or look at wikipedia: the articles about the "Peano axioms," "Negative and non-negative numbers," "Rational numbers," and "Dedekind cuts" will keep you busy for a while).

1 Basic properties of the real numbers

Fields

Definition 1.1. A *field* is a set F with operations "addition" and "multiplication" satisfying the following:

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- If a and b are in F, then so are $a+b$ and ab. (This is often phrased as, "The set F is **closed** under the operations of addition and multiplication.")
- Addition and multiplication are each **associative** (that is, $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c$ for all $a,b,c \in F$) and **commutative** (that is, $a+b=b+a$ and $ab=ba$ for all $a,b \in F$), and together they are **distributive**: $a(b+c)=ab+ac$ for all $a,b,c \in F$.
- F contains distinct elements called 0 and 1 satisfying:

$$a+0=a \quad \text{and} \quad a \cdot 1 = a$$

  for all $a \in F$. (The technical terms for these are **identity** elements for addition (0) and multiplication (1).)
- For each $a \in F$, there exists an element $b \in F$ so that $a+b=0$. (b is called the **additive inverse**, or negative, of a, and is written $-a$.)
- For each $a \in F$ with $a \neq 0$, there exists an element $b \in F$ so that $ab=1$. (b is called the **multiplicative inverse**, or reciprocal, of a, and is written a^{-1}.)

For example, the set $\mathbf{Q}$ of rational numbers and the set $\mathbf{C}$ of complex numbers form fields. The integers $\mathbf{Z}$ and the non-negative integers $\mathbf{N}$ do not. See Section 2.1 of the book for some more information about fields. The important fact for this class is the following.

Theorem 1.2. *The set* $\mathbf{R}$ *of real numbers forms a field.*

We won't prove this theorem; it would take too much time. Here are some basic properties of addition and multiplication in any field:

Proposition 1.3. *If F is a field (for example, if $F=\mathbf{R}$), then the following properties hold for any elements a, b, and c of F:*

(a) *if $a=b$, then $-a=-b$ (**uniqueness of additive inverses**)*

(b) *if $a=b$ and $a \neq 0$, then $a^{-1}=b^{-1}$ (**uniqueness of multiplicative inverses**)*

(c) *if $a+b=a+c$, then $b=c$ (**cancellation**)*

(d) *if $ab=ac$ and $a \neq 0$, then $b=c$ (**cancellation**)*

(e) $a \times 0 = 0$

(f) *if $ab=0$, then $a=0$ or $b=0$*

(g) $-0=0$

(h) $-(-a)=a$

(i) $(-a)b=a(-b)=-(ab)$

(j) $(-a)(-b)=ab$

(k) $(-1)a=-a$

This is only a sample; you can prove lots of other similar formulas.

Proof. Try proving this yourself – see Exercise 1. Here are a few to get you started:

For part (a): suppose that $a = b$. Add $-a$ to both sides: $a + (-a) = b + (-a)$. By the definition of additive inverses, the left side is zero: $0 = b + (-a)$. Now add $-b$ to both sides: $-b + 0 = (-b) + (b + (-a))$. By the definition of zero (on the left side) and associativity and additive inverses (on the right side), this becomes $-b = 0 + (-a)$. Finally, by the definition of zero, this is $-b = -a$.

For part (c): if $a + b = a + c$, then add $-a$ to both sides of the equation:

$$-a + (a + b) = -a + (a + c).$$

Now use associativity to rewrite this as

$$(-a + a) + b = (-a + a) + c.$$

By the definition of additive inverses, this becomes $0 + b = 0 + c$, and by the defining property of zero, this gives us $b = c$, as desired.

For part (e): since $0 = 0 + 0$, we have

$$\begin{aligned} a \times 0 &= a \times (0 + 0) \\ &= a \times 0 + a \times 0, \end{aligned}$$

by distributivity. Add $-(a \times 0)$ to both sides (that is, use part (c) to "cancel $a \times 0$ from both sides"): the resulting equation is $0 = a \times 0$, as desired. □

Inequalities

Definition 1.4. A field F is *ordered* if it has an ordering $<$ so that:

- For all $a, b \in F$, exactly one of these holds:

  $$a < b, \quad a = b, \quad a > b.$$

  This is called the **trichotomy law**.

- For all $a, b, c \in F$, if $a < b$, then $a + c < b + c$.

- For all $a, b \in F$, if $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.

(This is slightly different from the definition in the book, but this version is equivalent to theirs.)

For example, $\mathbf{Q}$ is an ordered field, while $\mathbf{C}$ is not – see below. The important fact for this class is the following.

Theorem 1.5. *The set of real numbers* $\mathbf{R}$ *is an ordered field.*

Again, we won't prove this theorem. Here are some basic properties of inequalities in any ordered field; see Section 2.2 for similar items. Essentially, all of the familiar formulas and procedures from high school algebra regarding equalities, inequalities, and algebraic manipulations of such should follow from the definition of an ordered field.

Proposition 1.6. *If F is an ordered field (for example, if $F = \mathbf{R}$), then the following properties hold for any elements a, b, and c of F:*

(a) $a < b$ *if and only if* $0 < b - a$

(b) *if* $a < b$ *and* $b < c$*, then* $a < c$ *(**transitivity**)*

(c) *if* $b < a$*, then* $-a < -b$

(d) *if* $0 < a$*, then* $-a < 0$

(e) *if* $a \neq 0$*, then* $a^2 > 0$

(f) $0 < 1$

(g) *if* $a < b$ *and* $c > 0$*, then* $ac < bc$

(h) *if* $a < b$ *and* $c < 0$*, then* $ac > bc$

Note that this shows that the set $\mathbf{C}$ of complex numbers cannot be an ordered field: no matter how you might try to define $<$ for complex numbers, as long as it satisfies the definition of an ordered field, then it must have $1 > 0$, and therefore (by part (b)) $-1 < 0$. However, since $i^2 = -1 < 0$, part (d) will fail.

Proof. Part (a): if $a < b$, then add $-a$ to both sides. By the definition of an ordered field, this produces a valid inequality, $0 < b - a$. Therefore $a < b$ implies $0 < b - a$. For the other implication, if $0 < b - a$, then adding a to both sides gives $a < b$.

Part (b): if $a < b$ and $b < c$, then by part (a), $0 < b - a$ and $0 < c - b$. Since the sum of positives is again positive, we have

$$0 < (b - a) + (c - b).$$

Using associativity and other basic properties of addition, we can write this as $0 < c - a$. Therefore by part (a) again, $a < c$, as desired.

Try proving the rest yourself – see Exercise 2. □

If F is an ordered field, define *absolute value* as usual: for any element a, its absolute value is

$$|a| = \begin{cases} a & \text{if } 0 \leq a, \\ -a & \text{if } a < 0. \end{cases}$$

Proposition 1.7. *For any elements* a *and* b *in an ordered field,*

(a) $|ab| = |a||b|$

(b) $|a + b| \leq |a| + |b|$ *(the **triangle inequality**)*

(c) $\big||a| - |b|\big| \leq |a - b|$

Proof. You should prove parts (a) and (c) – see Exercise 3.

We prove part (b). There are three cases to consider: both a and b non-negative, one non-negative and one negative, both negative.

If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$ (by the definition of ordered field), so by the definition of absolute value we have

$$|a + b| = a + b = |a| + |b|.$$

If $a \geq 0$ and $b < 0$, then $a \geq -a$ and $-b > b$. Subtract b from both sides of the first inequality: $a - b \geq -a - b$. Add a to both sides of the second inequality: $a - b > a + b$. By the definition of absolute value, $|a|+|b|=a-b$. If $a+b$ is negative, then $|a+b|=-a-b$, while if $a+b$ is non-negative, then $|a+b|=a+b$, and the previous two sentences tell us that $a-b$ is at least as big as each of these. So no matter the sign of $a+b$, we have $|a|+|b|\geq|a+b|$.

(If $a<0$ and $b\geq 0$, then the above argument works equally well, of course.)

If $a<0$ and $b<0$, then $-a>0$ and $-b>0$, so $(-a)+(-b)>0$, so $-(a+b)>0$, so $a+b<0$. So by the definition of absolute value, we have

$$|a+b|=-(a+b)=-a-b=(-a)+(-b)=|a|+|b|.$$

□

Least upper bounds

Both **Q** and **R** satisfy all of the properties given so far. Now we find a property which distinguishes between them.

Definition 1.8. Suppose that F is an ordered field and S is a nonempty subset of F. An *upper bound* for S is any element M of F so that $x\leq M$ for all $x\in S$. A *least upper bound* for S is any element L of F which is an upper bound for S and which also has the property that every $a<L$ is not an upper bound for S: L is the smallest upper bound for S.

For example, if S is the set of integers which are less than π, then S is nonempty because $1\in S$, and S has an upper bound because $\pi>x$ for all $x\in S$. S also has a least upper bound, namely 3. If T is the set of numbers

$$T=\left\{-1,-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},-\frac{1}{5},\ldots\right\},$$

then T is visibly nonempty, and also T has an upper bound: any positive number is an upper bound, since all of the elements of T are negative. The least upper bound for T is 0. The set **Z** of all integers has no upper bound and has no least upper bound.

Definition 1.9. An ordered field F has the *least upper bound property* if any nonempty subset $S\subseteq F$ with an upper bound has a least upper bound in F.

For example, **Q** does not have the least upper bound property – see Theorem 1.14 below – while the field **R** does. Summarizing, we have the following properties of **R**. By using this theorem, we can prove almost everything we need for the rest of the course.

Theorem 1.10 (Fundamental properties of **R**)**.** *The set* **R** *of real numbers forms an ordered field* **R** *which has the least upper bound property and which contains* **Q** *as a subfield.*

In fact, **R** is essentially the only ordered field with the least upper bound property, so this completely characterizes **R**. We will not prove this, nor will we prove the theorem. Instead, we basically take it as a definition of **R** and use it to prove everything else that we need.

Now, the book uses a slightly different approach, using what they call the "axiom of continuity" instead of the least upper bound property. If you want to compare the book's approach to ours, keep reading; otherwise, feel free to skip ahead to Theorem 1.13.

Definition 1.11. An ordered field F satisfies the *axiom of continuity* if, whenever it is divided into two nonempty subsets L and R so that

- every element of F is in either L or R,

- if a is in L and b is in R, then $a < b$ (the elements of L are "to the left" of the elements of R),

then there is an element $c \in F$ so that if $a < c$, then $a \in L$ while if $a > c$, then $a \in R$. (The element c itself may be in L or it may be in R.)

The least upper bound property is much more standard, so that's what we're using. The book proves that the axiom of continuity implies the least upper bound property (Theorem II in Section 2.7); we should prove the other implication to demonstrate that the two are equivalent, so that we can use either one when describing the real numbers.

Theorem 1.12. *Suppose that F is an ordered field which has the least upper bound property. Then it satisfies the axiom of continuity.*

Proof. Suppose we have L and R satisfying the hypotheses in the axiom of continuity. We need to find the element c satisfying the conclusions of the axiom. Since any element of R is an upper bound for L, the set L is a nonempty subset of F with an upper bound. Therefore it has a least upper bound; call it c. If $b \in R$, then b is an upper bound for L, and therefore $c \leq b$. Therefore if $a < c$, a must be in L. Now suppose that $a > c$. Since c is an upper bound for L, a may not be in L, and therefore a must be in R. This verifies that c is the number required by the axiom of continuity. □

The following theorem is a useful consequence of either the least upper bound property or of the axiom of continuity.

Theorem 1.13 (The Archimedean property)**.** *If a and b are positive real numbers, there is a positive integer n so that $b < na$.*

Proof. See the book – Theorem I in Section 2.4 – for a proof using the axiom of continuity. To use the least upper bound property instead, see Exercise 7. □

Theorem 1.14. *The field* **Q** *of rational numbers does not have the least upper bound property.*

Proof. To prove this, we need to use the following observation from Section 2.5 of Taylor and Mann: suppose that a and b are real numbers with $a < b$. Then there are rational numbers and irrational numbers between a and b.

Now consider the set

$$S = \{q \in \mathbf{Q} : q < \sqrt{2}\}.$$

Then S has an upper bound: any rational number larger than $\sqrt{2}$ will do. For example, 17 is an upper bound for S. It has no *least* upper bound (in **Q**), though: if $q \in \mathbf{Q}$ with $q < \sqrt{2}$, then by the fact cited above, there is a rational number s with $q < s < \sqrt{2}$. Since $s < \sqrt{2}$, s is an element of S, and therefore q is not an upper bound for S: it's not bigger than the element s. If $q > \sqrt{2}$, then q is an upper bound, but it cannot be the least upper bound: again by the fact above, there is a rational number r with $\sqrt{2} < r < q$. Since $r > \sqrt{2}$, r is an upper bound for S, and since it's less than q, q cannot be the least upper bound. □

The aforementioned fact from Taylor and Mann is useful enough to state it on its own:

Proposition 1.15. *If a and b are real numbers satisfying $a < b$, then there are rational numbers and irrational numbers between a and b.*

Proof. See Exercise 4. □

Finally, we also note that we can prove analogous results about lower bounds and greatest lower bounds, and in general, facts about lower bounds and greatest lower bounds will follow from the corresponding fact for upper bounds and least upper bounds. For example, we have the following: any field which has the least upper bound property also has the "greatest lower bound property."

Proposition 1.16. *Suppose that F is an ordered field satisfying the least upper bound property, and suppose that S is a nonempty subset of F with a lower bound. Then S has a greatest lower bound.*

Proof. Suppose that M is a lower bound for S: $M \leq x$ for every $x \in S$. Define a set T by $T = \{y : -y \in S\}$: that is, T is the set of the negatives of S. By basic properties of inequalities, we see that $-M \geq -x$ for every $x \in S$, or equivalently, $-M \geq y$ for every $y \in T$. That is, $-M$ is an upper bound for T. Since S was nonempty, so is T. By the least upper bound property, T has least upper bound c. It is easy to check that $-c$ is the greatest lower bound for S. □

Exercise 1. Prove the rest of Proposition 1.3.

Exercise 2. Prove the rest of Proposition 1.6.

Exercise 3. Prove the rest of Proposition 1.7. See Exercise 5 on p. 75 of the book for one approach.

Exercise 4. Prove Proposition 1.15. (See Exercise 2 on p. 79.)

Exercise 5. Prove that the square root of 2 is irrational. (See Exercise 1 on p. 79.)

Exercise 6. Prove that between any two numbers a and b with $a < b$, there are integers m and n with $n > 0$ and $a < m/10^n < b$. (This is Exercise 3 on p. 79.) That is, between any two numbers, there is a number with terminating decimal expansion.

Exercise 7. Use the least upper bound property to prove Theorem 1.13 – see Exercise 4 on p. 82.

2 Sequences and their limits

Basics

See Section 1.62 of the book for more about this subject.

A *sequence* of real numbers is an ordered set of real numbers indexed by the positive integers (or sometimes by the non-negative integers or the integers bigger than k for some k). We can write the sequence as

$$s_1, s_2, s_3, \ldots, s_n, s_{n+1}, \ldots$$

or as $\{s_n\}$ or as $\{s_n\}_{n\geq 1}$ (if we want to be explicit about the indexing set). For example, if $s_n = 1/2^n$, then the sequence is

$$s_1 = \frac{1}{2}, s_2 = \frac{1}{4}, s_3 = \frac{1}{8}, \ldots,$$

or $\{1/2^n\}$ or $\{1/2^n\}_{n\geq 1}$.

Definition 2.1. Suppose that $\{s_n\}$ is a sequence of real numbers. We say that this sequence *converges* to the number A, and we write $\lim_{n\to\infty} s_n = A$, if for every $\varepsilon > 0$, there is an integer N so that whenever $n \geq N$, we have $|s_n - A| < \varepsilon$.

If the sequence fails to converge, we say that it *diverges*.

Note that A must be a real number, and ∞ is not a real number. So if a sequence goes to infinity, this is just a particular kind of divergence.

Proposition 2.2. *Let a be a real number and consider the sequence $\{s_n\}$ defined by $s_n = a^n$ for all $n \geq 1$.*

(a) If $|a| < 1$, then this sequence converges to 0.

(b) If $a = 1$, this sequence converges to 1.

(c) If $a = -1$, this sequence diverges.

(d) If $|a| > 1$, this sequence diverges.

Proof. Part (a). If $a = 0$, then $a^n = 0$ for all n, so the sequence is

$$0,0,0,0,\dots.$$

This certainly looks like it should converge to 0, but let's prove it using the definition. Fix $\varepsilon > 0$. We need to find an N so that for all $n \geq N$, we have $|s_n - A| < \varepsilon$. In this case, $s_n = 0$ and $A = 0$, so we need $0 < \varepsilon$. But ε is positive by assumption. Therefore we pick N to be 1: for all $n \geq 1$, it is true that $|s_n - 0| < \varepsilon$, so the sequence converges to 0.

Now assume that $0 < |a| < 1$. Let $b = |a|$ and let $h = (1-b)/b$; then $h > 0$ and $b = 1/(1+h)$.

We want to show that $\lim_{n\to\infty} a^n = 0$, so fix $\varepsilon > 0$. We want to find N so that $n \geq N$ implies that $|a^n| < \varepsilon$, or equivalently, since $|a| = 1/(1+h)$,

$$\frac{1}{\varepsilon} < (1+h)^n.$$

I claim that for all $n \geq 1$, $(1+h)^n \geq 1+hn$ – this is an easy induction proof. (You should fill in the details in Exercise 8.) Now, since $h > 0$, we can appeal to the Archimedean property (Theorem 1.13) to conclude that there is a positive integer N so that $Nh > 1/\varepsilon - 1$, or equivalently, $1+Nh > 1/\varepsilon$. Therefore for all $n \geq N$, we have

$$(1+h)^n \geq 1+hn \geq 1+Nh > \frac{1}{\varepsilon}.$$

This means that $\lim_{n\to\infty} a^n = 0$.

Part (b). This is easy: since $1^n = 1$ for all n, we can use the same proof as in the case when $a = 0$.

Part (c). The sequence in question is

$$-1,1,-1,1,-1,1,\dots.$$

This doesn't appear to converge to anything, but we need to prove that. More precisely, we need to show this: for any number A, this sequence fails to converge to A. The definition of converging is:

$$\exists A \forall \varepsilon > 0 \exists N \forall n \geq N : |s_n - A| < \varepsilon.$$

We need to show that this fails, so we need to show the following:

$$\forall A \exists \varepsilon > 0 \forall N \exists n \geq N : |s_n - A| \geq \varepsilon.$$

Fix a real number A, let $\varepsilon = 1/2$ and fix an integer N. If $A < 0$, then find an even integer $n \geq N$: then $(-1)^n = 1$, and we have $|1-A| > 1 > \varepsilon$, as desired. If $A \geq 0$, then choose an odd integer $n \geq N$: then $(-1)^n = -1$ and $|-1-A| \geq 1 > \varepsilon$, as desired.

Part (d). This is similar to the previous part. Fix a real number A, let $\varepsilon = 1/2$, and fix an integer N. Note that since $|a| > 1$, then $|a^n| > 1$ for all n.

If $A < 0$, then find an even integer $n \geq N$: then $a^n = |a|^n > 1$, and we have

$$|a^n - A| > |a^n| > 1 > \varepsilon,$$

as desired. If $A \geq 0$, then choose an odd integer $n \geq N$: then $a^n = -|a|^n$ and again we have

$$|a^n - A| > |a^n| > 1 > \varepsilon,$$

as desired. □

We actually proved the following "obvious" fact in the previous proof, in the case $c = 0$. The proof generalizes easily.

Proposition 2.3. *If $\{s_n\}$ is a constant sequence, that is, there is a number c so that $s_n = c$ for all n, then $\lim_{n\to\infty} s_n = c$*

The book alludes to analogues of Theorems X, XI, and XII. Here they are.

Theorem 2.4. *(a) Let $\{s_n\}$ be a sequence of real numbers, and suppose that $\{s_n\}$ converges to A. If A is positive, then there is an integer N so that for all $n \geq N$, the number s_n is positive.*

(b) Let $\{s_n\}$ be a sequence of real numbers, and suppose that $\{s_n\}$ converges to A. Suppose that there is a real number M and an integer N so that $s_n \leq M$ for all $n \geq N$. Then $A \leq M$.

(c) Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be sequences of real numbers, and suppose that there is an integer N so that $r_n \leq s_n \leq t_n$ for all $n \geq N$. If $\lim_{n\to\infty} r_n = A = \lim_{n\to\infty} t_n$, then $\lim_{n\to\infty} s_n = A$.

Proof. Do it yourself; this is Exercise 9. □

Part (c) is often called the "squeeze theorem". Note that part (b) applies with strict inequalities, if you're careful:

Corollary 2.5. *Let $\{s_n\}$ be a sequence of real numbers, and suppose that $\{s_n\}$ converges to A. Suppose that there is a real number M and an integer N so that if $n \geq N$, then $s_n < M$. Then $A \leq M$.*

Proof. If $s_n < M$ for all $n \geq N$, then certainly $s_n \leq M$ for all $n \geq N$. So by the theorem, we have $A \leq M$. □

Note that if $s_n < M$ for all N, then you *cannot* conclude that $A < M$. You can only conclude that $A \leq M$: if $s_n = 1 - 1/n$ for all $n \geq 1$, then $s_n < 1$ for all n, but $\lim_{n\to\infty} s_n = 1$.

The book also mentions "general theorems about sums, products and quotients":

Theorem 2.6. *Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers.*

(a) If $\lim s_n = A$ and $\lim t_n = B$, then $\lim(s_n + t_n) = A + B$.

(b) If $\lim s_n = A$ and $\lim t_n = B$, then $\lim(s_n t_n) = AB$.

(c) If $\lim s_n = A$ and $\lim t_n = B$, and if $B \neq 0$, then $\lim(s_n/t_n) = A/B$.

The hypotheses here are important: you must know that each sequence $\{s_n\}$ and $\{t_n\}$ converges independently. If either one diverges, then the sum, product, or quotient may converge or diverge, depending on the situation.

Proof. Part (a) is straightforward. Parts (b) and (c) are a bit more complicated, but not too bad. Try them yourself: this is Exercise 10. $\square$

We also have the following theorem.

Theorem 2.7. *Suppose that $\{s_n\}$ is a sequence of real numbers converging to A. If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function, then the sequence $\{f(s_n)\}$ converges to $f(A)$:*

$$\lim_{n\to\infty} f(s_n) = f(A).$$

We won't prove this because we aren't dealing with the precise definition of "continuity" in this course: you'll have to wait until Math 328. You may use this theorem freely, though, along with the fact that essentially all of the standard functions are continuous wherever they're defined.

For example,

$$\lim_{n\to\infty} \sin(5 + (1/2)^n) = \sin(5).$$

Since $0 < 1/2 < 1$, we know that $\lim_{n\to\infty} (1/2)^n = 0$. The sequence $\{s_n\}$ defined by $s_n = 5$ for all n converges to 5, so the sequence $5 + (1/2)^n$ converges to $5 + 0 = 5$ by the theorem about sums of sequences. Since the sine function is continuous, we get the desired limit.

You might want to look at Example 3 in Section 1.62 for another application of some of the theorems from this section.

Bounded monotone sequences

A sequence $\{s_n\}$ is *monotonically increasing* if it satisfies this condition:

$$s_1 \leq s_2 \leq s_3 \leq s_4 \leq \dots,$$

that is, $s_n \leq s_{n+1}$ for all n. *Monotonically decreasing* is defined similarly. A sequence is *monotone* if it is either monotonically increasing or monotonically decreasing.

A sequence $\{s_n\}$ is *bounded above* if there is an M so that $s_n \leq M$ for all n, and similarly for *bounded below*.

Here is a useful theorem:

Theorem 2.8. *If $\{s_n\}$ is monotonically increasing and bounded above, then $\{s_n\}$ converges.*

Before proving this, we offer a property of least upper bounds.

Lemma 2.9. *Let S be a nonempty set and let A be a number. Then A is the least upper bound of S if and only if it satisfies these two properties:*

- *$x \leq A$ for all $x \in S$, and*
- *for every $\varepsilon > 0$, there is an $x \in S$ with $A - \varepsilon < x$.*

Proof. Suppose that A is the least upper bound of S. Then A is an upper bound, so $x \leq A$ for all $x \in S$. Fix $\varepsilon > 0$. Since $A - \varepsilon$ is strictly less than A, then $A - \varepsilon$ is not an upper bound for S, so there must be an $x \in S$ with $A - \varepsilon < x$.

To prove the converse, suppose that A satisfies the two properties in the statement of the lemma. The first property tells us that A is an upper bound of S. To show that A is the *least* upper bound, we have to show that no number smaller than A is an upper bound for S, so suppose that $B < A$. Let $\varepsilon = A - B$, so that $\varepsilon > 0$ and $B = A - \varepsilon$. By the second property, there is an element $x \in S$ with $x > A - \varepsilon = B$; therefore B is not an upper bound for S. □

Proof of Theorem 2.8. Assume that $s_n \leq M$ for all n. Let S be the set of numbers $\{s_1, s_2, s_3, \dots\}$. This set may be finite – for example, if $s_n = (-1)^n$, then $S = \{1, -1\}$ – or infinite, but it is certainly nonempty. It also has an upper bound, namely M. Therefore, since $\mathbf{R}$ has the least upper bound property, the set S has a least upper bound, say A. We claim that $\lim_{n\to\infty} s_n = A$.

Fix $\varepsilon > 0$. We need to show that there is an N so that if $n \geq N$, then $|s_n - A| < \varepsilon$, or equivalently, $A - s_n < \varepsilon$, or equivalently, $A - \varepsilon < s_n$. By the lemma, there is an N so that $s_N > A - \varepsilon$. For all n with $n \geq N$, we have $s_n \geq s_N$, and therefore if $n \geq N$, then $s_n > A - \varepsilon$. This finishes the proof. □

In fact, we have proved the following refinement of the theorem:

Corollary 2.10. *If $\{s_n\}$ is monotonically increasing and bounded above, then $\{s_n\}$ converges to the least upper bound of the set $\{s_1, s_2, \dots\}$.*

Here is an obvious variant.

Corollary 2.11. *If $\{s_n\}$ is monotonically decreasing and bounded below, then $\lim_{n\to\infty} s_n$ exists and equals the greatest lower bound of the set $\{s_1, s_2, s_3, \dots\}$.*

Proof. Exercise. □

Example 4 in Section 1.62 of Taylor & Mann is a straightforward example of the theorem. Another extremely important application is essentially the one in Example 5 of the book: define a sequence $\{s_n\}$ by

$$s_n = \sum_{i=0}^{n} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots \frac{1}{n!}.$$

This is clearly monotonically increasing, and Taylor and Mann show that it is bounded above by 3. (They actually consider a slightly different sequence – each term in theirs is missing the $i = 0$ term of the sum, and hence is one less than the corresponding term here – but their proof works equally well here.) Therefore the sequence converges to some number $A \leq 3$.

In Example 6 on p. 63, another very important example, Taylor and Mann show that the sequence $\{s_n\}$ defined by

$$s_n = \left(1 + \frac{1}{n}\right)^n$$

is also increasing and bounded above, again by 3. They define the number e to be the limit: e is defined by

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Exercise 27 below relates these two important examples to each other: it asks you to show that

$$\lim_{n\to\infty} \left(\sum_{i=0}^{n} \frac{1}{i!}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

and therefore both limits are equal to e.

Exercise 8. Show that for any real number h with $0 \leq h < 1$ and any positive integer n, $(1+h)^n \geq 1+hn$. (Use induction on n.)

Exercise 9. Prove Theorem 2.4.

Exercise 10. Prove Theorem 2.6. See Section 1.64 for pointers.

Exercise 11. Exercise 1 on p. 65

Exercise 12. Exercise 2 on p. 65

Exercise 13. Exercise 5 on p. 65

Exercise 14. Exercise 6 on p. 65

Exercise 15. Exercise 9 on p. 65

Exercise 16. Exercise 11 on p. 65

Exercise 17. Exercise 14 on p. 66

Exercise 18. Exercise 16 on p. 66

Exercise 19. Exercise 17 on p. 66

Exercise 20. Prove Corollary 2.11 from Corollary 2.10 by considering the sequence $\{t_n\}$ defined by $t_n = -s_n$.

Exercise 21. Exercise 5 on p. 82.

Exercise 22. Exercise 6 on p. 82.

Exercise 23. Exercise 19 on p. 66

Exercise 24. Exercise 20 on p. 66

Exercise 25. Exercise 21 on p. 66

Exercise 26. Exercise 22 on p. 66

Exercise 27. Exercise 24 on p. 66.