Mathematics 327 Final Exam Name: ______________________

June 8, 2009

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. Provide reasons for all of your answers.

1. (10 points) For which integers a does the following series converge? For which integers a does it diverge?

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} a^n$$

2. (10 points) Abel's test says:

   Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent, and that $b_n > 0$ and $b_n \geq b_{n+1}$ for all $n \geq 0$. Then $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

   Prove this.

3. (As in the text book, "log" means the natural log.)

(a) (5 points) Does the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converge or diverge?

(b) (5 points) Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$ converge or diverge?

(c) (5 points) For which real numbers x does the series $\sum_{n=1}^{\infty} \frac{x^n}{3n}$ converge, and for which does it diverge?

4. (15 points) Define $f(x)$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

for all real numbers x.

(a) Find a simple expression for $f(x)$ when $x \neq 0$. (Hint: factor out x^2 and use a geometric series.)

(b) What is $f(0)$? Does $f(x)$ have any discontinuities? Can you deduce anything about uniform convergence?

(c) Show that if a is any positive real number, then the series converges uniformly on the interval $[a, \infty)$.

5. (10 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ for $0 \leq x \leq 2\pi$.

(a) Does the series converge uniformly to $f(x)$?

(b) Is the equality

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{\sin nx}{n^2} \right)$$

valid for all x in $[0, 2\pi]$? (Or as the book phrases it, can $f'(x)$ be calculated for each x in the specified interval by differentiating the series for $f(x)$ term by term?)