Mathematics 327 Final Exam Name: Answers
June 8, 2009

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. Provide reasons for all of your answers.

1. (10 points) For which integers a does the following series converge? For which integers a does it diverge?

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} a^n$$

Solution: Use the ratio test. Write the nth term as u_n; then the ratio $|u_{n+1}/u_n|$ is

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{(n+1)^{n+1}|a|^{n+1}}{(n+1)!} \frac{n!}{n^n|a|^n} = \frac{(n+1)^n|a|}{n^n} = \left(1 + \frac{1}{n}\right)^n |a|.$$

As n goes to infinity, this approaches $e|a|$. Therefore if $|a| < 1/e$, this ratio is less than 1 and the series converges. If $|a| > 1/e$, it diverges. More explicitly, the series converges if $a = 0$, and it diverges for all integers a with $|a| \geq 1$.

2. (10 points) Abel's test says:

Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent, and that $b_n > 0$ and $b_n \geq b_{n+1}$ for all $n \geq 0$. Then $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

Prove this.

Solution: I'm going to apply Dirichlet's test. The a_n's satisfy the conditions of that test: the requirement is that the partial sums $\sum_{i=0}^{n} a_i$ be bounded, and since the series $\sum a_i$ converges, the sequence of partial sums converges, and hence they are bounded.

The b_n's, though, don't necessarily satisfy the condition that $b_n \to 0$. However, since the b_n's are decreasing and bounded below, the limit $\lim b_n$ exists, so let $b = \lim b_n$ and for each n, let $c = b_n - b$. Then $c_n \geq 0$ and $c_n \geq c_{n+1}$ for all n, and also $c_n \to 0$ as $n \to \infty$. Therefore the a_n's and the c_n's satisfy the conditions for Dirichlet's test, so the series $\sum a_n c_n$ converges. Plug in $c_n = b_n - b$; then we get

$$\sum a_n b_n = \sum a_n c_n + \sum a_n b.$$

Each sum on the right side converges; therefore the one on the left does as well.

(Note that the comparison test requires that all of the terms in the series be non-negative. We don't know anything about the signs of the series $\sum a_n$, so we can't use a comparison test here, at least not in any simple way.)

3. (As in the text book, "log" means the natural log.)

(a) (5 points) Does the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converge or diverge?

Solution: It converges. I'll verify this with the integral test: the series converges if and only if the improper integral

$$\int_2^\infty \frac{1}{x(\log x)^2} dx$$

converges. Do this integral with a substitution: let $u = \log x$, so $du = dx/x$. Then the integral equals

$$\int_{\log 2}^\infty \frac{du}{u^2} = \lim_{a\to\infty} \left.\frac{-1}{u}\right|_{\log 2}^{a} = \frac{1}{\log 2}.$$

Since the integral converges, so does the series.

(b) (5 points) Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$ converge or diverge?

Solution: It converges. I'll verify this using a comparison test. For $n \geq 3$, we have $\log n \geq \log 3 > 1$. Therefore for $n \geq 3$, we have $\frac{1}{n^{\log n}} < \frac{1}{n^{\log 3}}$. Since $\log 3$ is bigger than 1, the series $\sum \frac{1}{n^{\log 3}}$ converges; therefore the original series converges as well.

(c) (5 points) For which real numbers x does the series $\sum_{n=1}^{\infty} \frac{x^n}{3n}$ converge, and for which does it diverge?

Solution: Let's use the root test: let $u_n = x^n/3n$, and then

$$\sqrt[n]{|u_n|} = \frac{|x|}{(3n)^{1/n}}.$$

As $n \to \infty$, $n^{1/n}$ goes to 1, as does $3^{1/n}$; therefore $\sqrt[n]{|u_n|}$ goes to $|x|$. So if $|x| < 1$, the series converges and if $|x| > 1$, it diverges. What if $|x| = 1$? If $x = 1$, we have a harmonic series, which diverges. If $x = -1$, we have a series which converges by the alternating series test. Summarizing: if $-1 \leq x < 1$, the series converges; otherwise, it diverges.

The ratio test works just as well.

4. (15 points) Define $f(x)$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

for all real numbers x.

(a) Find a simple expression for $f(x)$ when $x \neq 0$. (Hint: factor out x^2 and use a geometric series.)

Solution: Note that if $x \neq 0$, then $1 + x^2 > 1$, so $\frac{1}{1+x^2} < 1$. We have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} \\ &= x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2}\right)^n = x^2 \frac{1}{1 - \frac{1}{1+x^2}} \quad \text{(geometric series)} \\ &= x^2 \frac{1+x^2}{1+x^2-1} = x^2 \frac{1+x^2}{x^2} \\ &= 1 + x^2. \end{aligned}$$

(b) What is $f(0)$? Does $f(x)$ have any discontinuities? Can you deduce anything about uniform convergence?

Solution: Clearly $f(0) = 0$: just plug $x = 0$ into the series defining $f(x)$. (The answer for part (a) explicitly says $x \neq 0$, so we can't plug $x = 0$ into that to find $f(0)$.) Therefore $f(x)$ has a discontinuity at 0: when $x \neq 0$, $f(x) = 1 + x^2$, and as x approaches zero, this approaches 1. Therefore, the series cannot converge uniformly on any interval containing 0.

(c) Show that if a is any positive real number, then the series converges uniformly on the interval $[a, \infty)$.

Solution: Fix a positive number a. Then for all $x \in [a, \infty)$, since $x \geq a$, then $\frac{1}{1+x^2} \leq \frac{1}{1+a^2}$. Since a is positive, $1/(1+a^2) < 1$, so the geometric series $\sum 1/(1+a^2)^n$ converges. By the Weierstrass M-test, the series defining $f(x)$ converges uniformly.

5. (10 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ for $0 \le x \le 2\pi$.

(a) Does the series converge uniformly to $f(x)$?

Solution: Yes: for all n and x, we have

$$\left| \frac{\sin nx}{n^2} \right| \le \frac{1}{n^2},$$

and so by the Weierstrass M-test, the series converges uniformly.

(b) Is the equality

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{\sin nx}{n^2} \right)$$

valid for all x in $[0, 2\pi]$? (Or as the book phrases it, can $f'(x)$ be calculated for each x in the specified interval by differentiating the series for $f(x)$ term by term?)

Solution: No. Consider the series

$$\sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{\sin nx}{n^2} \right) = \sum_{n=1}^{\infty} \frac{n \cos nx}{n^2} = \sum_{n=1}^{\infty} \frac{\cos nx}{n}.$$

When $x = 0$, this becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series, and hence diverges. So the series of derivatives doesn't converge at all, let alone uniformly.