Mathematics 327 Midterm Exam Name: Answers
November 3, 2010

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. (10 points) Let $s_1, s_2, s_3, \ldots$ be a sequence of real numbers converging to A. Suppose that for some real number M and some integer $K \geq 1$, $s_n \leq M$ for all $n \geq K$. Prove that $A \leq M$.

   (Don't just cite a theorem – prove this straight from the definitions.)

   Solution: Suppose that $A > M$. Let $\varepsilon = A - M$; this is positive since $A > M$. For every $n \geq K$, $s_n \leq M < A$, and so $|A - s_n| = A - s_n \geq \varepsilon$. This means that $\{s_n\}$ cannot converge to A.

   That is, under the assumption that $s_n \leq M$ for all $n \geq K$, we have proved that if $A > M$, then $\{s_n\}$ cannot converge to A. Taking the contrapositive gives the desired result.

   Alternatively, if $\{s_n\}$ converges to A, then for every $\varepsilon > 0$, there is an N so that $|s_n - A| < \varepsilon$ whenever $n \geq N$. Therefore if $n \geq \max(N, K)$, we have $A < s_n + \varepsilon \leq M + \varepsilon$. Therefore for all $\varepsilon > 0$, we must have $A < M + \varepsilon$. The only way for this to be true for all $\varepsilon > 0$ is if $A \leq M$. (If $A > M$, then let $\varepsilon = A - M$: we get $A = M + \varepsilon$, and in particular $A \not< M + \varepsilon$.)

2. (10 points) Let $s_n = \dfrac{n^2+n-3}{2n^2+2}$. What is $\lim_{n\to\infty} s_n$? Prove that your answer is correct using the definition of the limit (and all of the properties of ordered fields with the least upper bound property, but for full credit, avoid using theorems about limits).

Solution: First, we can tell that the limit is $1/2$; the issue is how to prove it. So we focus our attention on $|s_n - 1/2|$; we want to show that this can be made less than ε for any positive ε.

So fix $\varepsilon > 0$ and consider $|s_n - 1/2|$; we have

$$\begin{aligned} |s_n - 1/2| &= \left| \frac{n^2+n-3}{2n^2+2} - \frac{1}{2} \right| \\ &= \left| \frac{(n^2+n-3)-(n^2+1)}{2n^2+2} \right| \\ &= \left| \frac{n-4}{2n^2+2} \right|. \end{aligned}$$

We would like this to be less than ε. There are at least two good approaches: we can solve the inequality

$$\left| \frac{n-4}{2n^2+2} \right| < \varepsilon,$$

which (when $n \geq 4$, so that we can remove the absolute values) is equivalent to

$$n-4 < \varepsilon(2n^2+2), \quad \text{or} \quad 0 < \varepsilon(2n^2+2) - n + 4, \quad \text{or} \quad 0 < 2\varepsilon n^2 - n + (4+2\varepsilon).$$

As long as n is larger than the larger of the two roots of the quadratic on the right, then this inequality will hold. Actually, there are two cases: either the quadratic has no roots, in which case (since it's upward-opening) it's always positive, so any value of n will work (but remember that we assumed that $n \geq 4$); or it has at least one root which we can find using the quadratic formula, and we want n to be larger than that root. So for the two cases, we let N be the smallest integer at least as big as

$$\begin{cases} \max(4, \frac{1+\sqrt{1-32\varepsilon-16\varepsilon^2}}{4\varepsilon}) & \text{if at least one root,} \\ 4 & \text{if no roots.} \end{cases}$$

Alternatively, we can simplify the expression $\left|\frac{n-4}{2n^2+2}\right|$, using inequalities: assuming that $n \geq 4$,

$$\left| \frac{n-4}{2n^2+2} \right| = \frac{n-4}{2n^2+2} \leq \frac{n}{2n^2+2} \leq \frac{n}{2n^2} = \frac{1}{2n}.$$

Thus if $1/2n < \varepsilon$, then the same will be true of the quantity we actually care about. This is easy to solve: it holds if $1/2\varepsilon < n$. So let N be the smallest integer larger than both 4 (we assumed this earlier) and $1/2\varepsilon$. We have showed that if $n \geq N$, then $1/2n < \varepsilon$, so

$$|s_n - 1/2| = \left| \frac{n-4}{2n^2+2} \right| < \varepsilon.$$

This proves that the sequence converges to $1/2$.

3. Let S be the subset of the real line consisting of rational numbers q such that $1 \leq q^2 \leq 3$. Answer the following, giving brief justifications for your answers.

   (a) (5 points) Find the least upper bound and greatest lower bound of S.

   Solution: First, the set in question is

   $$S = \left((-\sqrt{3}, -1] \cap \mathbf{Q}\right) \cup \left([1, \sqrt{3}) \cap \mathbf{Q}\right).$$

   The least upper bound is $\sqrt{3}$ and the greatest lower bound is $-\sqrt{3}$. Certainly $\sqrt{3}$ is an upper bound: if q is any rational number satisfying $q^2 \leq 3$, then $q \leq \sqrt{3}$. Also, no smaller number is an upper bound: if $1 < y < \sqrt{3}$, then there is a rational q between y and $\sqrt{3}$, and q will satisfy $q^2 < 3$, and so will be in S. A similar argument holds for the greatest lower bound.

   (b) (5 points) Find all of the accumulation points of S.

   Solution: The set of accumulation points is

   $$[-\sqrt{3}, -1] \cup [1, \sqrt{3}].$$

   If y is not in either of these closed intervals, then it's easy to find an ε-neighborhood of y which doesn't intersect S. If y is in one of these closed intervals, then for any $\varepsilon > 0$, the neighborhood $(y - \varepsilon, y + \varepsilon)$ will intersect S. For example if $y = \sqrt{3}$, then there will be rationals between $y - \min(\varepsilon, \sqrt{3} - 1)$ and y, and those rationals will be in the set S.

   (c) (5 points) Is S open?

   Solution: No, S is not open. S contains the point 1, but no ε-neighborhood of 1 is contained in S, because any such neighborhood will contain some points strictly between -1 and 1, and S contains no such points. (In fact, for any element q of S, any neighborhood of q will contain irrationals, but since S contains only rationals, that neighborhood can't be in S.)

   (d) (5 points) Is S closed?

   Solution: No, S is not closed. Here are two reasons: first, it doesn't contain $\sqrt{3}$, but $\sqrt{3}$ is an accumulation point. By one of our theorems, we know that if a set doesn't contain all of its accumulation points, it cannot be closed. Second, the point $\sqrt{3}$ is in the complement S^c, but no ε-neighborhood of $\sqrt{3}$ is contained in S^c: any such neighborhood will contain rationals to the left of $\sqrt{3}$, and those are in S. (In fact, any irrational y between 1 and $\sqrt{3}$ will be in the complement, but arguing as in (c), any neighborhood of y will contain some rationals between 1 and $\sqrt{3}$, and hence will not be contained in S^c.)