Mathematics 327 Final Exam Name: Answers

December 15, 2010

Instructions: This is a closed book exam, no notes (except what I've provided) or calculators allowed. Please turn off all cell phones, pagers, etc.

Provide full explanations and justifications for all of your answers. Simplify your answers when possible.

1. (7 points) Suppose that the numbers b_n are positive, decreasing, and go to zero as $n \to \infty$. Consider the series

$$b_0 + b_1 - b_2 - b_3 + b_4 + b_5 - b_6 - b_7 + \dots$$

The pattern is that there are two positive terms, then two negative, then two positive, etc. Use Dirichlet's test to prove that this series converges.

Solution: We let $a_0 = a_1 = 1$, $a_2 = a_3 = -1$, $a_4 = a_5 = 1$, etc. The number b_n already satisfy condition (a) of Theorem 6 on the note sheet, so to apply Dirichlet's test, we just need to verify condition (a): find some constant M independent of n such that

$$|a_0 + a_1 + \cdots + a_n| \leq M$$

for all values of n. The partial sums $a_0 + a_1 + \cdots + a_n$ take on the values 1 (when $n = 0$), 2 (when $n = 1$), 1 ($n = 2$), 0 ($n = 3$), and then repeat. So for all n,

$$|a_0 + a_1 + \cdots + a_n| \leq 2.$$

Therefore the numbers a_n satisfy condition (b), so by Dirichlet's test, the series converges.

2. (a) (7 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converge conditionally, converge absolutely, or diverge?

Solution: It converges conditionally. First, the terms alternate in sign, they are decreasing in absolute value ($1/\sqrt{n+1} \leq 1/\sqrt{n}$ for all n), and they approach zero, so by the alternating series test, the series does converge. Second, consider the series of absolute values, $\sum_{n=1}^{\infty} 1/\sqrt{n}$. Since $1/\sqrt{n} \geq 1/n$ for all $n \geq 1$ and since the series $\sum 1/n$ diverges (harmonic series), then the series $\sum 1/\sqrt{n}$ diverges by the comparison test. (Alternatively, $\sum 1/\sqrt{n}$ is a p-series with $p = 1/2$, and therefore diverges.) Therefore the original series does not converge absolutely. Since it converges but not absolutely, it converges conditionally.

(b) (7 points) Consider the series $\displaystyle\sum_{n=0}^{\infty} (n+2)^{10}/(n+3)^{10+q}$. For what values of q does it converge? [Hint: compare to a suitable p-series.]

Solution: Compare to the series $\sum 1/n^q$; this converges if and only if $q > 1$. I'll use the second of the convergence tests, looking at the limit of ratios of the terms

$$\frac{\frac{(n+2)^{10}}{(n+3)^{10+q}}}{\frac{1}{n^q}} = \frac{(n+2)^{10} n^q}{(n+3)^{10+q}} = \frac{(n+2)^{10}}{(n+3)^{10}} \frac{n^q}{(n+3)^q} = \frac{(n+2)^{10}}{(n+3)^{10}} \frac{1}{(1+3/n)^q}.$$

As n goes to infinity, this goes to 1. Therefore the series $\sum (n+2)^{10}/(n+3)^{10+q}$ converges if and only if the series $\sum 1/n^q$ does, and this converges if and only if $q > 1$. (It is not good enough to just show that the series converges if $q > 1$; for full credit, you must also explain why it doesn't converge if $q \leq q$. The "if and only if" statements in my solution take care of that.)

3. (a) (7 points) For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \left(2 + \cos \frac{n\pi}{3}\right)^n \left(\frac{x}{2}\right)^n$ converge?

Solution: Use the root test: the nth root of the absolute value of the nth term is

$$\left| \left(2 + \cos \frac{n\pi}{3}\right) \left(\frac{x}{2}\right) \right|.$$

Since $\cos(n\pi/3)$ varies between -1 and 1, the first factor can be at most 3. Therefore if $3|x|/2 < 1$, that is, if $|x| < 2/3$, the series will converge. On the other hand, if $3|x| > 1$, then whenever $n/3$ is even, $\cos(n\pi/3)$ will equal 1, so the nth root will be larger than 1. This will happen for infinitely many values of n, so by the root test, the series will diverge. Summarizing so far, the series will converge if $|x| < 2/3$ and diverge if $|x| > 2/3$. What if $|x| = 2/3$? In this case, the nth term is

$$\left(2 + \cos \frac{n\pi}{3}\right)^n \frac{1}{(\pm 3)^n}.$$

These terms don't go to zero, so the series can't converge. Therefore the series will converge if and only if $|x| < 2/3$.

(Just as in the previous problem, for full credit you need to give an "if and only if" answer.)

(b) (7 points) Does the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converge or diverge?

Solution: It converges. You can see this using the integral test, the ratio test, or the root test.

Integral test: let $f(x) = xe^{-x^2}$. Then $f(x)$ is continuous and positive when $x > 0$. By computing the derivative, you can see that $f(x)$ is non-increasing as long as $x > 1/\sqrt{2}$. Therefore in the range under consideration ($x \geq 1$), $f(x)$ is positive and non-increasing, so the test applies. The test says that the series converges if and only if the improper integral $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx$ converges. Make the substitution $u = x^2$ and do the integral; it is easy to check that the integral converges.

I'll leave the details for the ratio test and the root test for you.

4. Let $f_n(x) = nx/(nx+1)$ for $n \geq 1$ and let $f(x) = \lim_{n\to\infty} f_n(x)$.

(a) (2 points) Find the value of $f(x)$ for every x.

Solution: If $x = 0$, then $f(0) = 0$. If $x \neq 0$, then $f(x) = 1$.

(b) (3 points) Is convergence uniform on $[0, 1]$?

Solution: No. The functions $f_n(x)$ are continuous but the limit function is discontinuous at $x = 0$; therefore convergence can't be uniform on any interval containing 0.

(c) (3 points) Fix $r > 0$. Is convergence uniform on $[r, \infty)$?

Solution: Yes. Fix $\varepsilon > 0$. We want find N so that for all $x \geq r$ and all $n \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Since $f(x) = 1$ for all $x > 0$, we can rewrite $|f_n(x) - f(x)|$ as

$$|f_n(x) - f(x)| = |\frac{nx}{nx+1} - \frac{nx+1}{nx+1}| = |\frac{1}{nx+1}|.$$

Since $x \geq r$, $1/(nx+1) \leq 1/(nr+1)$. We want this to be less than ε, so solve for n: let $N = 1/r(1/\varepsilon - 1)$. Then simple algebra shows that if $n \geq N$, then

$$\frac{1}{nx+1} \leq \frac{1}{nr+1} < \varepsilon.$$

Therefore convergence is uniform.

5. Let $f(x) = \sum_{n=0}^{\infty} (2x)^n$.

(a) (3 points) What is a simple formula for $f(x)$ when $|x| < 1/2$? Fix r with $0 < r < 1/2$ and prove uniform convergence of this series on the interval $[-r, r]$.

Solution: This is a geometric series with ratio $2x$, so it equals $1/(1-2x)$. This is valid whenever $|2x| < 1$, i.e., whenever $|x| < 1/2$.
If $|x| \leq r$, then $(2x)^n \leq (2r)^n$, so let $M_n = (2r)^n$. Since $r < 1/2$, the series $\sum M_n$ converges, so by the Weierstrass M-test, the series defining $f(x)$ converges uniformly on the interval $[-r, r]$.

(b) (4 points) Now compute the integral $\int_0^{3/8} f(x)\,dx$ using the series and using the simple formula from part (a) to come up with a series and its sum.

Solution: Since we have uniform convergence on $[-r, r]$ for any $r < 1/2$, we have uniform convergence on any subinterval, and in particular on $[0, 3/8]$. Therefore

$$\int_0^{3/8} \frac{1}{1-2x}\,dx = \sum_{n=0}^{\infty} \int_0^{3/8} (2x)^n\,dx.$$

The left side is a simple integral; it equal $\ln 2$. The right side is also easy, it equals

$$\sum_{n=0}^{\infty} 2^n \frac{(3/8)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(3/4)^{n+1}}{2(n+1)}.$$

Therefore

$$\ln 2 = \sum_{n=0}^{\infty} \frac{(3/4)^{n+1}}{2(n+1)}.$$