Mathematics 135 Quiz 4 Name: Answers

February 4, 2010

Instructions: This is a closed book quiz, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. Find the solution to the initial value problem

$$y' + \frac{1}{x}y = x^2, \quad y(1) = 0.$$

Solution: This is a first order linear equation. The integrating factor is therefore

$$e^{\int 1/x\,dx} = e^{\ln|x|} = |x| = x.$$

(Since we are interested in the solution where $x_0 = 1$, which is positive, we may assume, at least at the start, that $x > 0$. Then, since the equation doesn't make sense when $x = 0$, it doesn't make sense for our solution to extend to non-positive numbers. So throughout, we may assume that $x > 0$.)

So multiply the equation by x and integrate:

$$\begin{aligned} xy' + y &= x^3 \\ \frac{d}{dx}(xy) &= x^3 \\ xy &= \int x^3\,dx = \frac{1}{4}x^4 + c \\ y &= \frac{1}{4}x^3 + cx^{-1}. \end{aligned}$$

Now apply the initial condition $y(1) = 0$:

$$0 = \frac{1}{4} + c, \quad \text{so} \quad c = 1.$$

The solution is therefore

$$y = \frac{1}{4}c^3 - \frac{1}{4}x^{-1}.$$

2. Consider the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 1.$$

Starting with $y_0(x) = 1$, compute the first Picard iterate $y_1(x)$.

Solution: The formula for $y_1(x)$ is

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt.$$

In this case, $x_0 = 0$, $y_0 = 1$, and $y_0(t)$ is the constant function 1. Also, $f(t,y) = t^2 + y^2$. So the formula becomes

$$y_1(x) = 1 + \int_0^x (t^2 + 1)\, dt = 1 + x + \frac{x^3}{3}.$$