**Mathematics 135 Midterm 2** Name: Answers

February 26, 2010

**Instructions**: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. As usual, justify all of your answers.

1. (10 points) Let $c$ be a constant, and let $y(t)$ be the solution to the initial value problem

$$y'' - 4y = \delta(t-1) + c\delta(t-3), \quad y(0) = 0,\ y'(0) = 0.$$

For what value(s) of $c$ does $y(t)$ satisfy $\lim_{t\to\infty} y(t) = 0$?

**Solution:** Let's find $y(t)$ first. Take Laplace transforms:

$$(s^2 - 4)\mathcal{L}(y) = e^{-s} + ce^{-3s},$$

so

$$\mathcal{L}(y) = \left(e^{-s} + ce^{-3s}\right)\frac{1}{s^2-4} = \left(e^{-s} + ce^{-3s}\right)\frac{1}{(s-2)(s+2)}$$
$$= \left(e^{-s} + ce^{-3s}\right)\left(\frac{1/4}{s-2} - \frac{1/4}{s+2}\right).$$

Therefore

$$y = \frac{1}{4}u(t-1)\left(e^{2(t-1)} - e^{-2(t-1)}\right) + \frac{c}{4}u(t-3)\left(e^{2(t-3)} - e^{-2(t-3)}\right).$$

So for $t > 3$,

$$y = \frac{1}{4}\left(e^{2(t-1)} - e^{-2(t-1)}\right) + \frac{c}{4}\left(e^{2(t-3)} - e^{-2(t-3)}\right).$$

For this to go to zero as $t \to \infty$, the coefficient of $e^{2t}$ must be zero. That coefficient is

$$\frac{1}{4}e^{-2} + \frac{c}{4}e^{-6}.$$

Set this equal to zero and solve for $c$: $c = -e^4$.

2. (10 points) The solution to the initial value problem

$$(1+x^2)y'' + y' - 2y = 0, \quad y(0) = 1,\ y'(0) = 2,$$

can be expressed in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find the recursion formula for the coefficients. Also find $a_0$, $a_1$, $a_2$ and $a_3$.

**Solution:** If $y = \sum a_n x^n$, then $y' = \sum n a_n x^{n-1}$ and $y'' = \sum n(n-1)a_n x^{n-2}$, all sums starting at $n = 0$. Therefore the differential equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Reindex all of the sums so that they are in terms of $x^n$:

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=-1}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

In the first sum, the $n = -1$ and $n = -2$ terms are zero, so it can start at $n = 0$ instead of $n = -2$; similarly for the third sum. So we get

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1)a_{n+2} + n(n-1)a_n + (n+1)a_{n+1} - 2a_n\Big] x^n = 0.$$

The coefficient of each power of $x$ must be zero, so we get the recursion formula by setting the coefficient of $x^n = 0$: for each $n \geq 0$,

$$(n+2)(n+1)a_{n+2} + (n^2 - n - 2)a_n + (n+1)a_{n+1} = 0.$$

Note that $n^2 - n - 2 = (n-2)(n+1)$, so we can cancel a factor of $n+1$, resulting in the recursion formula:

$$a_{n+2} = \frac{-(n-2)a_n - a_{n+1}}{n+2}.$$

From the initial conditions, $a_0 = 1$ and $a_1 = 2$. From the recursion relation, $a_2 = 0$ and $a_3 = 2/3$.

3. (10 points) Find the general solution to the equation

$$y'' + 2y' + 2y = 2\cos t.$$

**Solution:** The characteristic equation is $m^2 + 2m + 2 = 0$, which has roots $m = -1 \pm i$. Therefore

$$y_c = c_1 e^{-t} \sin t + c_2 e^{-t} \cos t.$$

For $y_p$, try $y_p = A\cos t + B\sin t$. Plug this in and solve for $A$ and $B$; you should get $A = 2/5$ and $B = 4/5$. Therefore the general solution is

$$y = c_1 e^{-t} \sin t + c_2 e^{-t} \cos t + \frac{2}{5}\cos t + \frac{4}{5}\sin t.$$

4. Consider the equation

$$y'' - 4y' + 4y = f(t).$$

(a) (2 points) What is $y_c$, the general solution to the associated homogeneous equation?

> **Solution:** The characteristic equation is $m^2 - 4m + 4 = 0$, which has $m = 2$ as a double root. So
> $$y_c = c_1 e^{2t} + c_2 t e^{2t}.$$

(b) (4 points) Let $f(t) = (t^2 - 7)\cos 2t$. According to the method of undetermined coefficients, what should you try for $y_p$? (Don't solve for the coefficients: just give me the form for $y_p$.)

> **Solution:** Since $f(t)$ has no terms in common with $y_c$, $y_p$ takes the form
> $$y_p = (At^2 + Bt + C)\cos 2t + (Dt^2 + E + F)\sin 2t.$$

(c) (4 points) Let $f(t) = te^{2t}$. According to the method of undetermined coefficients, what should you try for $y_p$? (Don't solve for the coefficients: just give me the form for $y_p$.)

> **Solution:** In this case, $f(t)$ is a polynomial multiplied by a summand in $y_c$, so the initial guess $(At + B)e^{2t}$ won't work. Multiply it by $t^2$ instead:
> $$y_p = (At^3 + Bt^2)e^{2t}.$$

5. (5 points) Let $f(t)$ be a continuous function, and suppose that $y_1(t)$ and $y_2(t)$ are solutions to the equation

$$y'' + 4y = f(t)$$

satisfying the initial conditions

$$y_1(0) = 2,\ y_1'(0) = 2, \quad y_2(0) = 2,\ y_2'(0) = 0.$$

Find $y_1(t) - y_2(t)$.

**Solution:** Since this is a linear equation, the general solution has the form $y = y_c + y_p$. Therefore the difference between any two solutions is a solution to the associated homogeneous equation. In this case, since the characteristic equation is $m^2 + 4 = 0$ with roots $m = \pm 2i$, that means that the difference $y_{\text{diff}} = y_1 - y_2$ takes the form

$$y_{\text{diff}} = c_1 \cos 2t + c_2 \sin 2t.$$

Furthermore, we know that $y_{\text{diff}}$ satisfies the initial conditions

$$y_{\text{diff}}(0) = y_1(0) - y_2(0) = 0, \quad y'_{\text{diff}}(0) = y_1'(0) - y_2'(0) = 2.$$

Therefore

$$y_{\text{diff}} = \sin 2t.$$

6. (5 points) Find the general solution to the equation

$$x^2 y' + xy = 3x^3.$$

**Solution:** This is a first order linear equation. Put it in standard form by dividing by the coefficient of $y'$: $y' + (1/x)y = 3x$. Then the integrating factor is $e^{\int P(x)dx}$, where $P(x) = 1/x$. This integrating factor is therefore equal to $x$, and the equation becomes

$$xy' + y = 3x^2.$$

Integrate:

$$xy = x^3 + c, \quad \text{so} \quad y = x^2 + cx^{-1}.$$