Mathematics 135 Midterm 1 Name: Answers
January 27, 2009

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. As usual, justify all of your answers.

1. (10 points) Evaluate $\lim_{x\to 0} \dfrac{\cos x - \cos 2x}{x \sin 4x}$.

> **Solution:** Note that the numerator and denominator both go to zero, so we could use L'Hôpital's rule; the derivatives look a little messy, so instead I'll replace all of the functions by their Taylor series:
>
> $$\begin{aligned}\lim_{x\to 0} \frac{\cos x - \cos 2x}{x \sin 4x} &= \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)}{x\left(4x - \frac{(4x)^3}{3!} + \dots\right)} \\ &= \lim_{x\to 0} \frac{\frac{3x^2}{2!} + \dots}{4x^2 - \dots} = \frac{3}{8}.\end{aligned}$$
>
> (When dealing with limits as $x \to 0$ of quotients of polynomials or power series, everything is controlled by the terms in top and bottom of smallest order, so I've discarded all higher order terms.)
>
> If you use L'Hôpital's rule instead, you just have to differentiate top and bottom twice to get (of course) the same answer.

2. (10 points) Evaluate the integral $\displaystyle\int_0^1 \frac{dx}{x^{2/5}}$ or explain why you can't.

> **Solution:** Because of the denominator, this is improper when $x = 0$, so we attempt to compute the integral as a limit:
>
> $$\begin{aligned}\int_0^1 \frac{dx}{x^{2/5}} &= \lim_{a\to 0^+} \int_a^1 x^{-2/5}\, dx \\ &= \lim_{a\to 0^+} \left[\frac{5}{3} x^{3/5}\right]_a^1 \\ &= \lim_{a\to 0^+} \left(\frac{5}{3} - \frac{5}{3} a^{3/5}\right).\end{aligned}$$
>
> As a approaches zero, this has a limit, $5/3$, so the integral converges, and its value is $5/3$.

3. (10 points) Use the formula $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$, valid for $-1 < x < 1$, to compute the Taylor series for $\tanh^{-1} x$. What is the interval of convergence for the series?

Solution: The formula $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$ gives us $\tanh^{-1} x = \int \frac{1}{1-x^2}$. Also, when $-1 < x < 1$, $\frac{1}{1-x^2}$ is the sum of a geometric series with ratio x^2. Therefore when $-1 < x < 1$,

$$\begin{aligned} \tanh^{-1} x &= \int \frac{1}{1-x^2}\, dx \\ &= \int \left(\sum_{k=0}^{\infty} x^{2k}\right) dx \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \end{aligned}$$

The ratio test tells us that the radius of convergence is 1. The series diverges when $x = 1$ (compare to harmonic series) and also when $x = -1$ (because it's just -1 times the series when $x = 1$). So the interval of convergence is $(-1, 1)$.

(When you apply the theorem about term-by-term of integration of series, it only tells you the *radius* of convergence: you have to deal with the endpoints separately.)

4. (10 points) For each integer $n \geq 1$, let $a_n = 2\ln(3n-1) - \ln(2n^2 + 2n + 3)$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, what is the limit?

Solution: First we rewrite a_n:

$$\begin{aligned} a_n &= \ln\left((3n-1)^2\right) - \ln\left(2n^2 + 2n + 3\right) \\ &= \ln\left(\frac{9n^2 - 6n + 1}{2n^2 + 2n + 3}\right). \end{aligned}$$

Since ln is a continuous function, we have

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \ln\left(\frac{9n^2 - 6n + 1}{2n^2 + 2n + 3}\right) \\ &= \ln\left(\lim_{n\to\infty} \frac{9n^2 - 6n + 1}{2n^2 + 2n + 3}\right). \end{aligned}$$

The limit inside is $9/2$, so the sequence converges to $\ln(9/2)$.

5. (a) (5 points) Does the series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converge or diverge?

Solution: The ratio test seems like the most appropriate tool. Every term is positive, so we don't need to worry about conditional convergence, and we also don't need any absolute value signs.

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(k+1)!k^k}{k!(k+1)^{k+1}} \\ &= \frac{k^k}{(k+1)^k} \\ &= \left(\frac{k}{k+1}\right)^k \\ &= \frac{1}{\left(1+\frac{1}{k}\right)^k}. \end{aligned}$$

As $k \to \infty$, the denominator approaches e, so the whole thing approaches $1/e$. Since this is less than 1, the series converges.

(b) (5 points) Does the series $\sum_{k=1}^{\infty} (-1)^k \frac{k+2}{k^3+k}$ converge absolutely, converge conditionally, or diverge?

Solution: Take absolute values. Then the numerator is less than or equal to $2k$ (as long as $k \geq 2$) while the denominator is larger than k^3, so for $k \geq 2$, we have

$$\frac{k+2}{k^3+k} \leq \frac{2k}{k^3} = 2\frac{1}{k^2}.$$

The series $\sum 2/k^2$ converges – it's a constant multiple of a p-series with $p = 2$, which converges since $p > 1$. So by the comparison test, the absolute value of the original series converges. Thus the series converges absolutely.

Alternatively, the limit comparison test is a good one to use here: we know that $\sum 1/k^2$ converges, and the limit $(1/k^2)/((k+2)/(k^3+k))$ is 1, so after taking absolute values, the series converges. Therefore the original series converges absolutely.