Three old Math 135 midterms

(1) Evaluate $\lim_{x\to 0} \dfrac{x\sin(x^2) - \sin(x^3)}{\sin(x^7)}$ in any way you wish.

(2) Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx$ or explain why you can't.

(3) Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k \ln(k+1)}{k} x^k$.

(a) Find its radius of convergence.

(b) Find its interval of convergence.

(c) For what values of x the series absolutely convergent? For what values of x is the series conditionally convergent?

(4) Consider the sequence $\{a_k\}$ defined by $a_0 = 0$, $a_{n+1} = 1 + m\,a_n$, where m is a real number with $|m| < 1$. Does the sequence converge? Explain your answer. If the sequence does converge, what is its limit?

(5) Find the first three non-zero terms in the series expansion of $\arcsin(x)$ about $x = 0$.

Hint: *Recall that* $\arcsin(x) = \int_0^x (1-t^2)^{-1/2} dt$. *If you wish, you may use the binomial expansion of* $(1+x)^{-1/2}$.

---

(1) Evaluate $\lim_{x\to 0} \dfrac{\cosh x - \cos x}{\sin x^2}$ in any way you wish.

(2) Is the series $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k^3 + \ln k + 1}$ absolutely convergent, conditionally convergent, or divergent? Justify your answer

(3) Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{2x}{1-x^2}\,dx$ or explain why you can't.

(4) Give the Taylor series about 0 of the function $f(x) = \displaystyle\int_0^x \sin(t^2)\,dt$. For what values of x does the series converge to $f(x)$? Justify your answer.

(5) Consider the sequence $\{a_k\}$ defined by $a_0 = 0$,

$$a_0 = 1, \quad a_{n+1} = 1 - a_n/2 \text{ for } n = 0, 1, 2, \ldots.$$

Show that the sequence converges to 2/3.

---

(1) Evaluate the limit $\lim_{x\to 0} \dfrac{\cosh x - \cos x}{x^2}$.

(2) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ or explain why you can't.

(3) Does the series $1 + \dfrac{1\cdot 2}{1\cdot 3} + \dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \dfrac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \ldots$ converge?

(4) Test the following two series for (i) absolute and (ii) conditional convergence.

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^k \frac{k+2}{k^2+k}$$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{k^k}{k!}$$

(5) Let S be the set of numbers of the form $(-1)^{n^2} \dfrac{n+1/n!}{n+1}$ for $n \geq 2$. Explain why S does or does not have a least upper bound. If it has a least upper bound, what is it? Answer the same question about greatest lower bounds.