Math 135: Homework 10

Due Thursday, March 11

(1) A vector-valued function $\mathbf{G}$ is called an *antiderivative* for $\mathbf{f}$ on $[a, b]$ provide that $\mathbf{G}$ is continuous on $[a, b]$ and $\mathbf{G}'(t) = \mathbf{f}(t)$ for all $t \in (a, b)$.

(a) Show that if $\mathbf{f}$ is continuous on $[a, b]$ and $\mathbf{G}$ is an antiderivative for $\mathbf{f}$ on $[a, b]$ then

$$\int_a^b \mathbf{f}(t)\, dt = \mathbf{G}(b) - \mathbf{G}(a)$$

(b) Show that if $\mathbf{f}$ is continous on $[a, b]$ and $\mathbf{F}$ and $\mathbf{G}$ are antiderivatives for $\mathbf{f}$ on $[a, b]$ then

$$\mathbf{F} = \mathbf{G} + \mathbf{C}$$

for some constant vector $\mathbf{C}$.

(2) Three objects move in space according to the equations

$$\mathbf{r} = \mathbf{r}_1(t) \quad \mathbf{r} = \mathbf{r}_2(t) \quad \text{and } \mathbf{r} = \mathbf{r}_3(t)\,,$$

where t denotes time. Let $A(t)$ denote the area of the triangle formed by the three objects. Suppose that

$$\begin{array}{lll} \mathbf{r}_1(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} & \mathbf{r}_2(0) = \mathbf{i} + \mathbf{j} - \mathbf{k} & \mathbf{r}_3(0) = \mathbf{k} \\ \mathbf{r}_1'(0) = \mathbf{i} & \mathbf{r}_2'(0) = \mathbf{j} & \mathbf{r}_3'(0) = \mathbf{k} \end{array}$$

Compute $A'(0)$.

(3) Let $\mathbf{r} = \mathbf{r}(s)$ be the arc length parametrization of a simple curve. Suppose that $\|\mathbf{r}(s)\| = 1$ for all s and that $\mathbf{r}(s)$ has continuous first and second derivatives.

(a) Show that the three vector-valued functions $\mathbf{r} = \mathbf{r}(s)$, $\mathbf{T} = \mathbf{T}(s) =: \frac{d\mathbf{r}(s)}{ds}$, and $\mathbf{U} = \mathbf{U}(s) =: \mathbf{r}(s) \times \mathbf{T}(s)$ form an oriented frame (i.e. that they are mutually orthogonal unit vectors and that the triple scalar product $(\mathbf{r} \times \mathbf{T}) \cdot \mathbf{U}$ is positive).

(b) Show that there is a scalar function $\beta(s)$ such that the following equations are satisfied:

$$\frac{d\mathbf{r}}{ds} = \mathbf{T}\,, \quad \frac{d\mathbf{T}}{ds} = -\mathbf{r} + \beta\mathbf{U}\,, \quad \frac{d\mathbf{U}}{ds} = -\beta\mathbf{T}\,.$$

(c) Now let $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ denote the Frenet frame for the curve. Show that

$$\mathbf{N} = \frac{-\mathbf{r} + \beta\mathbf{U}}{\sqrt{1 + \beta^2}}$$

and from that conclude that $\kappa = \sqrt{1 + \beta^2}$.

(d) Finally, use part (c) to find a formula for the torsion τ of the curve in terms of β and its derivative β'. From this, conclude that $\tau(s) = 0$ if and only if $\beta'(s) = 0$.