Mathematics 135 Final Exam

Name: Answers

March 15, 2010

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. When you have a choice, real solutions are preferable to complex ones. As usual, justify all of your answers.

Q: What do you get when you cross an orange with a banana?
A: Orange banana $\sin\theta$.

1. (5 points) Suppose that a curve C is specified parametrically by a vector function $\mathbf{r}(t)$. What is the formula for the curvature κ? For full credit, do *not* express your answer in terms of an arc length parametrization.

Solution: Let $\mathbf{T}(t)$ be the unit tangent vector, defined by

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}.$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

2. (10 points) Find the radius of convergence for the series

$$\sum_{k=0}^{\infty} \frac{(2k)^k}{k!} x^{2k}.$$

Solution: Use the ratio test: if b_k denotes the kth term in the series, then

$$\begin{aligned}
\left|\frac{b_{k+1}}{b_k}\right| &= \left|\frac{2^{k+1}(k+1)^{k+1}k!x^{2(k+1)}}{2^k k^k (k+1)! x^{2k}}\right| \\
&= \left|\frac{2(k+1)^{k+1}x^2}{k^k(k+1)}\right| \\
&= 2x^2\left(\frac{k+1}{k}\right)^k \\
&= 2x^2\left(1+\frac{1}{k}\right)^k.
\end{aligned}$$

Since $(1+1/k)^k \to e$, the limit of this ratio is $2x^2e$. This is less than 1, guaranteeing convergence, if $x^2 < 1/2e$, which means that $|x| < 1/\sqrt{2e}$. Thus the radius of convergence is $1/\sqrt{2e}$.

3. (10 points) Define a curve by

$$\mathbf{r}(t) = \cos 3t\,\mathbf{i} + t\,\mathbf{j} - \sin 3t\,\mathbf{k}.$$

Find its *binormal* $\mathbf{B}(t)$: this is defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, where $\mathbf{T} = \mathbf{T}(t)$ is the unit tangent vector and $\mathbf{N} = \mathbf{N}(t)$ is the principal normal vector.

(Check your work: what should the dot product $\mathbf{T} \cdot \mathbf{B}$ be? How about $\mathbf{N} \cdot \mathbf{B}$? How about $\mathbf{B} \cdot \mathbf{B}$?)

Solution: First we compute $\mathbf{T} = \mathbf{r}' / \|\mathbf{r}'\|$:

$$\mathbf{r}'(t) = -3\sin 3t\,\mathbf{i} + \mathbf{j} - 3\cos 3t\,\mathbf{k},$$

which has length $\|\mathbf{r}\|'(t) = \sqrt{10}$. Therefore

$$\mathbf{T} = \frac{1}{\sqrt{10}}\left(-3\sin 3t\,\mathbf{i} + \mathbf{j} - 3\cos 3t\,\mathbf{k}\right).$$

Next, $\mathbf{N} = (d\mathbf{T}/dt) / \|d\mathbf{T}/dt\|$, and

$$d\mathbf{T}/dt = \frac{1}{\sqrt{10}}\left(-9\cos 3t\,\mathbf{i} + 9\sin t\,\mathbf{k}\right).$$

This has length $9/\sqrt{10}$, so

$$\mathbf{N} = -\cos 3t\,\mathbf{i} + \sin t\,\mathbf{k}.$$

Finally, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$:

$$\mathbf{B} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-3\sin 3t}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{-3\cos 3t}{\sqrt{10}} \\ -\cos 3t & 0 & \sin t \end{pmatrix} = \frac{1}{\sqrt{10}}\left(\sin t\,\mathbf{i} + 3\,\mathbf{j} + \cos 3t\,\mathbf{k}\right).$$

4. (a) (3 points) Find the general solution to the equation

$$y'''' - y = 0.$$

Solution: The characteristic equation is $m^4 - 1 = 0$. This has roots 1, -1, i, $-i$ (for example, factor it as $(m^2 - 1)(m^2 + 1) = 0$). These are distinct, so the general solution is

$$y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x.$$

(b) (7 points) Find the general solution to the equation

$$y'' - 3y' - 10y = 14e^{-2t}.$$

Solution: First find the solution y_c to the associated homogeneous equation. The characteristic equation is $m^2 - 3m - 10 = 0$, which has roots -2 and 5 (either factor it or use the quadratic formula). So we get

$$y_c = c_1 e^{-2t} + c_2 e^{5t}.$$

Since the right side of the equation is one of the pieces of y_c, we have to try

$$y_p = Ate^{-2t}$$

for a particular solution. Plug this in and solve for A: $y_p' = e^{-2t}(A - 2At)$ and $y_p'' = e^{-2t}(-4A + 4At)$, so

$$y_p'' - 3y_p' - 10y_p = e^{-2t}\left(-4A + 4At - 3A + 6At - 10At\right) = e^{-2t}(-7A).$$

This is supposed to equal $14e^{-2t}$, so $A = -2$. Therefore the general solution is

$$y = y_c + y_p = c_1 e^{-2t} + c_2 e^{5t} - 2te^{-2t}.$$

5. (10 points) The initial value problem

$$(x+1)^2 y'' = x+3, \quad y(0)=0,\, y'(0)=0$$

has a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find a_n for $n \leq 3$ and find the general recursion relation, making sure to specify the values of n for which it is valid.

Solution: Let $y = \sum a_n x^n$, so $y' = \sum n a_n x^{n-1}$ and $y'' = \sum n(n-1)a_n x^{n-2}$. Then

$$\begin{aligned}(x+1)^2 y'' &= (x^2+2x+1)y'' = (x^2+2x+1)\sum n(n-1)a_n x^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.\end{aligned}$$

Reindex the second and third sums so they are in terms of x^n, and set equal to $x+3$:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2(n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = x+3,$$

or

$$\sum_{n=0}^{\infty} \left(n(n-1)a_n + 2(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2}\right) x^n = x+3.$$

Equating coefficients of x^n on the left and the right, we have

$$\begin{aligned} a_2 &= 3, && (n=0) \\ 4a_2 + 6a_3 &= 1, && (n=1) \\ n(n-1)a_n + 2(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} &= 0. && (n \geq 2)\end{aligned}$$

The initial conditions give $a_0 = 0 = a_1$. The equation for $n=0$ tells us that $a_2 = 3/2$. The equation for $n=1$ says that $a_3 = (1-4a_2)/6 = -5/6$. The equation for $n \geq 2$ gives the recursion relation, therefore valid for all $n \geq 2$:

$$a_{n+2} = -\frac{n(n-1)a_n + 2(n+1)n a_{n+1}}{(n+2)(n+1)}.$$

6. (a) (8 points) Solve the initial value problem

$$y'' + 4y = f(t), \quad y(0) = 0, \, y'(0) = 0,$$

where

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \cos 2t & \text{if } 0 \leq t < 2\pi, \\ 0 & \text{if } t \geq 2\pi. \end{cases}$$

(b) (2 points) Sketch the solution.

Solution: We can write $f(t)$ using the Heaviside step function. Note also that with Laplace transform, what happens for $t < 0$ is irrelevant, so we don't need to worry about that part of the formula:

$$f(t) = \cos 2t - u(t - 2\pi) \cos 2t = \cos 2t - u(t - 2\pi) \cos 2(t - 2\pi).$$

Apply the Laplace transform to the differential equation, writing $Y for \mathcal{L}(y)$:

$$(s^2 + 4)Y = \frac{s}{s^2 + 4} - \frac{se^{-2\pi s}}{s^2 + 4},$$

so

$$Y = \frac{s}{(s^2 + 4)^2} - \frac{se^{-2\pi s}}{(s^2 + 4)^2}.$$

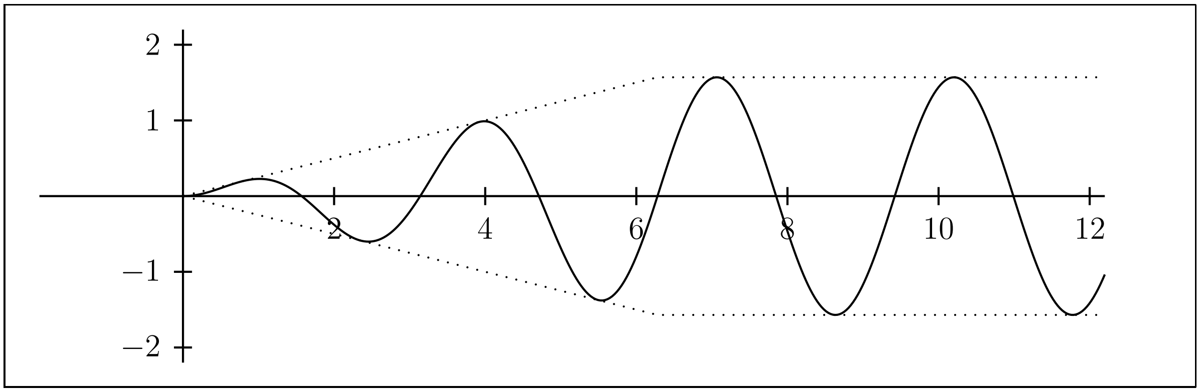
Therefore (using the line on the table for $\dfrac{2as}{(s^2 + a^2)^2}$ with $a = 2$):

$$\begin{aligned} y &= \frac{1}{4} t \sin 2t - u(t - 2\pi) \frac{1}{4}(t - 2\pi) \sin 2(t - 2\pi) \\ &= \frac{1}{4} t \sin 2t - u(t - 2\pi) \frac{1}{4}(t - 2\pi) \sin 2t \\ &= \frac{1}{4} t \sin 2t - u(t - 2\pi) \frac{1}{4} t \sin 2t + u(t - 2\pi) \frac{\pi}{2} \sin 2t. \end{aligned}$$

That is,

$$y = \begin{cases} \frac{1}{4} t \sin 2t & \text{if } 0 \leq t < 2\pi, \\ \frac{\pi}{2} \sin 2t & \text{if } t \geq 2\pi. \end{cases}$$

This looks like a sine curve, period π, with steadily increasing amplitude for $0 \leq t < 2\pi$ – resonance: it's oscillating between the lines $y = -t/4$ and $y = t/4$ – and then a sine curve, same period, with amplitude $\pi/2$ after that: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