1 Preliminaries

If $f(t)$ is defined on the interval $[0, \infty)$, then its Laplace transform is defined to be

$$F(s) = \mathcal{L}((t)) = \int_0^\infty e^{-st} f(t)\, dt,$$

as long as this integral is defined and converges. In particular, if f is of *exponential order* and is *is piecewise continuous*, the Laplace transform of $f(t)$ will be defined.

- f is of *exponential order* if there are constants M and c so that

  $$|f(t)| \leq Me^{ct}.$$

  Since the integral $\int_0^\infty e^{-st} Me^{ct}\, dt$ converges if $s > c$, by a comparison test, the integral defining the Laplace transform of $f(t)$ will converge.

- f is *piecewise continuous* if over each interval $[0, b]$, $f(t)$ has only finitely many discontinuities, and at each point a in $[0, b]$, both of the limits

  $$\lim_{t \to a^-} f(t) \quad \text{and} \quad \lim_{t \to a^+} f(t)$$

  exist – they need not be equal, but they must exist. (At the endpoints 0 and b, the appropriate one-sided limits must exist.)

2 Step functions

Define $u(t)$ to be the function

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Then $u(t)$ is called the *step function*: it jumps from 0 to 1 at $t = 0$. Note that for any number $a > 0$, the graph of the function $u(t-a)$ is the same as the graph of $u(t)$, but translated right by a: $u(t-a)$ jumps from 0 to 1 at $t = a$.

Proposition 1. *The Laplace transform of $u(t-a)$ is e^{-as}/s. If $f(t)$ is a function with Laplace transform $F(s)$, then*

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

What does the function $u(t-a)f(t-a)$ look like?

3 δ-functions

Proposition 2. *(a) Let ϵ be a positive number and consider the function $f_\epsilon(t)$ defined by*

$$f_\epsilon(t) = \begin{cases} 1/\epsilon & \text{if } 0 \leq t \leq \epsilon, \\ 0 & \text{if } t > \epsilon. \end{cases}$$

Then

$$\mathcal{L}(f_\epsilon(t)) = \frac{1 - e^{s\epsilon}}{s\epsilon}.$$

(b) "Define" the Dirac delta function *$\delta(t)$ to be*

$$\delta(t) = \lim_{\epsilon \longrightarrow 0} f_\epsilon(t).$$

Then $\delta(t) = 0$ except when $t = 0$, and it has the following properties with respect to integration: for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \qquad \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).$$

Therefore for any positive number a, we have $\mathcal{L}(\delta(t-a)) = e^{-as}$.

Now, $\delta(t)$ is not a function: the limit defining it doesn't exist when $t = 0$, for one thing. If there were a way to define it, then properties of integrals show that if $g(t)$ is any function with $g(t) = 0$ whenever $t \neq 0$, then $\int_a^b g(t) = 0$ for any a and b. Instead, $\delta(t)$ is what is called a *distribution*, and although it isn't a function, it can be treated like one in many ways. Really its defining property is that for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).$$