More About Taylor Polynomials

Suppose $f(x)$ has $n+1$ continuous derivatives, and let $P_n(x)$ be the nth Taylor polynomial of f (about $a = 0$). The estimate for the remainder $R_{n+1}(x) = f(x) - P_n(x)$ on p. 668 of Salas-Hille-Etgen (formula 11.5.3) can be restated as follows:

$$\text{If } |f^{(n+1)}(x)| \le C \text{ for } x \text{ in some interval } I \text{ containing } 0\text{, then} \quad |R_n(x)| \le \frac{C|x|^{n+1}}{(n+1)!} \text{ for } x \in I. \tag{1}$$

"Big O" notation: If $g(x)$ is a function defined near $x = 0$, and there is a constant C such that $|g(x)| \le C|x|^k$ for x near 0, we say that $g(x)$ is $O(x^k)$ (as $x \to 0$). O bears much the same relation to o as $\le$ does to $<$. That is, "$g(x) = o(x^k)$" means that $g(x) \to 0$ *faster than* x^k as $x \to 0$, whereas "$g(x) = O(x^k)$" means that $g(x) \to 0$ *at least as fast as* x^k as $x \to 0$.

With this notation, according to (1) we have $R_n(x) = O(x^{n+1})$, or

$$f(x) = P_n(x) + O(x^{n+1}). \tag{2}$$

Moreover P_n is the *only* polynomial of degree $\le n$ with this property. Indeed:

Proposition 1. *Suppose f has $n+1$ continuous derivatives, and suppose Q_n is a polynomial of degree $\le n$ such that $f(x) = Q_n(x) + O(x^{n+1})$ as $x \to 0$. Then Q_n is the nth Taylor polynomial of f.*

Proof: Let P_n be the nth Taylor polynomial of f. Subtracting the equations $f(x) - Q_n(x) = O(x^{n+1})$ and $f(x) - P_n(x) = O(x^{n+1})$, we obtain $P_n(x) - Q_n(x) = O(x^{n+1})$. In other words, if $P_n(x) = \sum_0^n a_k x^k$ and $Q_n(x) = \sum_0^n a_k x^k$,

$$(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = O(x^{n+1}). \tag{3}$$

Setting $x = 0$, we see that $a_0 - b_0 = 0$, or $a_0 = b_0$. This being so, if we divide (3) by x we get

$$(a_1 - b_1) + (a_2 - b_2)x + \cdots + (a_n - b_n)x^{n-1} = O(x^n).$$

Setting $x = 0$, we see that $a_1 = b_1$. Now we can divide (3) by x^2:

$$(a_2 - b_2) + (a_3 - b_3)x + \cdots + (a_n - b_n)x^{n-2} = O(x^{n-1}).$$

Setting $x = 0$ again, we get $a_2 = b_2$. Continuing inductively, we find that $a_k = b_k$ for all k, so $P_n = Q_n$. ∎

Proposition 1 is useful for calculating Taylor polynomials. It shows that using the formula $a_k = f^{(k)}(0)/k!$ is not the only way to calculate P_n; rather, if by *any* means we can find a polynomial Q_n of degree $\le n$ such that $f(x) = Q_n(x) + O(x^{n+1})$, then Q_n must be P_n. Here are two useful applications of this fact.

Taylor Polynomials of Products. Let P_n^f and P_n^g be the nth Taylor polynomials of f and g, respectively. Then

$$\begin{aligned} f(x)g(x) &= \left[P_n^f(x) + O(x^{n+1})\right]\left[P_n^g(x) + O(x^{n+1})\right] \\ &= \left[\text{terms of degree} \le n \text{ in } P_n^f(x)P_n^g(x)\right] + O(x^{n+1}). \end{aligned}$$

Thus, to find the nth Taylor polynomial of fg, simply multiply the nth Taylor polynomials of f and g together, discarding all terms of degree $> n$.

Example 1. What is the 6th Taylor polynomial of x^3e^x? Solution:

$$x^3e^x = x^3\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)\right] = x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{6} + O(x^7),$$

so the answer is $x^3 + x^4 + \frac{1}{2}x^5 + \frac{1}{6}x^6$.

Example 2. What is the 5th Taylor polynomial of $e^x \sin x$? Solution:

$$\begin{aligned} e^x \sin x &= \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right]\left[x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)\right] \\ &= x + x^2 + x^3\left[\frac{1}{2} - \frac{1}{6}\right] + x^4\left[\frac{1}{6} - \frac{1}{6}\right] + x^5\left[\frac{1}{24} - \frac{1}{12} + \frac{1}{120}\right] + O(x^6), \end{aligned}$$

so the answer is $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$.

Taylor Polynomials of Compositions. If f and g have derivatives up to order $n+1$ *and* $g(0) = 0$, we can find the nth Taylor polynomial of $f \circ g$ by substituting the Taylor expansion of g into the Taylor expansion of f, retaining only the terms of degree $\le n$. That is, suppose

$$f(x) = a_0 + a_1x + \cdots + a_nx^n + O(x^{n+1}).$$

Since $g(0) = 0$ and g is differentiable, we have $g(x) = O(x)$ and hence

$$f(g(x)) = a_0 + a_1g(x) + \cdots + a_ng(x)^n + O(x^{n+1}).$$

Now plug in the Taylor expansion of g on the right and multiply it out, discarding terms of degree $> n$.

Example 3. What is the 16th Taylor polynomial of e^{x^6}? Solution:

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3) \qquad \Longrightarrow \qquad e^{x^6} = 1 + x^6 + \frac{x^{12}}{2} + O(x^{18}),$$

so the answer is $1 + x^6 + \frac{1}{2}x^{12}$.

Example 4. What is the 4th Taylor polynomial of $e^{\sin x}$? Solution:

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + O(x^5)$$

since $\sin x = O(x)$. Now substitute $x - \frac{1}{6}x^3 + O(x^5)$ for $\sin x$ on the right and multiply out, throwing all terms of degree > 4 into the "$O(x^5)$" trash can:

$$e^{\sin x} = 1 + \left[x - \frac{x^3}{6}\right] + \frac{1}{2}\left[x^2 - \frac{x^4}{3}\right] + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5),$$

so the answer is $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4$.

Taylor Polynomials and l'Hôpital's Rule. Taylor polynomials can often be used effectively in computing limits of the form $0/0$. Indeed, suppose f, g, and their first $k-1$ derivatives vanish at $x = 0$, but their kth derivatives do not both vanish. The Taylor expansions of f and g then look like

$$f(x) = \frac{f^{(k)}(0)}{k!}x^k + O(x^{k+1}), \qquad g(x) = \frac{g^{(k)}(0)}{k!}x^k + O(x^{k+1}).$$

Taking the quotient and cancelling out $x^k/k!$, we get

$$\frac{f(x)}{g(x)} = \frac{f^{(k)}(0) + O(x)}{g^{(k)}(0) + O(x)} \to \frac{f^{(k)}(0)}{g^{(k)}(0)} \text{ as } x \to 0.$$

This is in accordance with l'Hôpital's rule, but the devices discussed above for computing Taylor polynomials may lead to the answer more quickly than a direct application of l'Hôpital.

Example 5. What is $\lim_{x\to 0}(x^2 - \sin^2 x)/x^2 \sin^2 x$? Solution:

$$\sin^2 x = \left[x - \frac{x^3}{6} + O(x^5)\right]^2 = x^2 - \frac{x^4}{3} + O(x^5),$$

so $x^2 \sin^2 x = x^4 + O(x^5)$, and

$$\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{\frac{1}{3}x^4 + O(x^5)}{x^4 + O(x^5)} = \frac{\frac{1}{3} + O(x)}{1 + O(x)} \to \frac{1}{3}.$$

Example 6. Evaluate

$$\lim_{x\to 1}\left[\frac{1}{\log x} + \frac{x}{x-1}\right].$$

Solution: Here we need to expand in powers of $x - 1$. First of all,

$$\frac{1}{\log x} - \frac{x}{x-1} = \frac{x - 1 - x\log x}{(x-1)\log x} = \frac{(x-1) - (x-1)\log x - \log x}{(x-1)\log x}.$$

Next, $\log x = (x-1) - \frac{1}{2}(x-1)^2 + O((x-1)^3)$, and plugging this into the numerator and denominator gives

$$\frac{(x-1) - (x-1)^2 - \left[(x-1) - \frac{1}{2}(x-1)^2\right] + O((x-1)^3)}{(x-1)^2 + O((x-1)^3)} = \frac{-\frac{1}{2} + O(x-1)}{1 + O(x-1)} \to -\frac{1}{2}.$$

Higher Derivative Tests for Critical Points. Recall that if $f'(a) = 0$, then $f(x)$ has a local minimum (resp. maximum) at $x = a$ if $f''(a) > 0$ (resp. $f''(a) < 0$). What happens if $f''(a) = 0$? Answer: The behavior of f near a is controlled by its first nonvanishing derivative at a.

Proposition 2. *Suppose f has k continuous derivatives near a, and $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$ but $f^{(k)}(a) \neq 0$. If k is even, f has a local minimum or maximum at a according as $f^{(k)}(a) > 0$ or $f^{(k)}(a) < 0$. If k is odd, f has neither a minimum nor a maximum at a.*

Proof: The $(k-1)$th Taylor polynomial of f about a is simply the constant $f(a)$ (all the other terms are zero), so Taylor's formula of order $k-1$ with the Lagrange form of the remainder R_k becomes

$$f(x) = f(a) + \frac{f^{(k)}(c)}{k!}(x-a)^k \text{ for some } c \text{ between } x \text{ and } a.$$

Now, if x (and hence c) is close to a, $f^{(k)}(c)$ is close to $f^{(k)}(a)$. In particular, it is nonzero and has the same sign as $f^{(k)}(a)$. On the other hand, $(x-a)^k$ is always positive if k is even but changes sign at $x = a$ if k is odd. Thus, if k is even, $f(x) - f(a)$ is positive or negative for x near a according to the sign of $f^{(k)}(a)$; but if k is odd, $f(x) - f(a)$ changes sign as x crosses a. The result follows. ∎