Problem 24, section 1.62

Let $b_n = \left(1+\frac{1}{n}\right)^n$ and let $a_p = \sum_{k=0}^{p} \frac{1}{k!}$. From Example 6 in the book, we know that the sequence $\{b_n\}$ is increasing and bounded above and that its limit is e; therefore $b_n < e$ for all n. Also from Example 6 (in particular, about a third of the way down on page 64), we know that $b_n < a_n$ for all n.

Therefore, we can follow the book's hint and try to conclude that $\lim b_n \le \lim a_n$ – I don't think this is obvious, but it can be justified using the definition of limit.[1] More importantly, though, we *don't actually need this* for the rest of the problem: all we need is that $b_n < a_n$ for all n.

Equation (1.62-3) says that

$$b_n = 1 + \frac{1}{1!} + \frac{1-\frac{1}{n}}{2!} + \cdots + \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}{n!}.$$

This is a sum with $n+1$ terms (terms with denominators $0!$, $1!$, $2!$, ..., $n!$). We could write it as

$$b_n = 1 + \sum_{k=1}^{n} \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!}.$$

For any p with $1 < p < n$, if we take the first $p+1$ terms, then the sum is smaller than b_n, and that sum is

$$1 + \frac{1}{1!} + \frac{1-\frac{1}{n}}{2!} + \cdots + \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{p-1}{n}\right)}{p!} = 1 + \sum_{k=1}^{p} \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!}.$$

Denote this sum by $c_{p,n}$. Fix p and let $n \to \infty$ here: the denominator of each fraction goes to 1, so the limit[2] is

$$\lim_{n\to\infty} c_{p,n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} = a_p.$$

[1] There are two possibilities: either $a < e$ or $a \ge e$. We want to rule out the first of these, so let's use contradiction: assume that $a < e$. Let $\varepsilon = (e-a)/2$, so that $e = a + 2\varepsilon$. Since $\lim b_n = e$, there is an N so that $|e - b_n| < \varepsilon$ for all $n \ge N$. Since $b_n < e$ for each n, we can rewrite this as $e - b_n < \varepsilon$, or $e - \varepsilon < b_n$. Therefore $a + \varepsilon < b_n < a_n$. This means that $|a_n - a| > \varepsilon$. Since a_n is farther than ε from a for all $n \ge N$, the limit $\lim a_n$ cannot equal a. This is a contradiction; therefore $a \ge e$.

[2] We can break this step down: for each positive integer i, the expression $1 - i/n$ goes to 1 as $n \to \infty$. Since the limit of a product is the product of the limits, then for each k,

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = 1,$$

and since the limit of a quotient is the quotient of the limits,

$$\lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} = \frac{1}{k!}.$$

Finally, since the limit of a sum is the sum of the limits, we have

$$\lim_{n\to\infty}\left(1 + \sum_{k=1}^{p} \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!}\right) = 1 + \sum_{k=1}^{p} \frac{1}{k!} = a_p.$$

Since $c_{p,n} < b_n$ for all p and n, and since $b_n < e$ for all n, we have that $c_{p,n} < e$ for all p and n. Therefore the analogue of Theorem XI for sequences says that $\lim c_{p,n} \leq e$, which means that $a_p \leq e$. Therefore we have (replacing p with n):

$$b_n < a_n \leq e \quad \text{for all } n.$$

Now apply the squeeze principle: since $\lim b_n = e$, we conclude that $a = e$, as desired.

Alternatively, once we know that $a_p \leq e$ for all p, then we apply the analogue of Theorem XI for sequences again to conclude that $a \leq e$. If we have also established that $a \geq e$, then these together imply that $a = e$.