Ordered fields (Taylor and Mann, §2.1–2.2)

**Definition 1.** A field is a set $F$ with operations “addition” and “multiplication” satisfying the following:

- If $a$ and $b$ are in $F$, then so are $a + b$ and $ab$.
- Addition and multiplication are each associative and commutative, and together they are distributive: $a(b + c) = ab + ac$ for all $a,b,c \in F$.
- $F$ contains elements called 0 and 1 satisfying:
  
  \[
  a + 0 = a \quad \text{and} \quad a \cdot 1 = a
  \]

  for all $a \in F$.
- For all $a \in F$, there exists an element $b \in F$ so that $a + b = 0$.
- For all $a \in F$ with $a \neq 0$, there exists an element $b \in F$ so that $ab = 1$.

**Example 2.** The set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$ each form fields. The integers $\mathbb{Z}$ and the non-negative integers $\mathbb{N}$ do not.

**Definition 3.** A field $F$ is ordered if it has an ordering $<$ so that:

- For all $a, b \in F$, exactly one of these holds:
  
  \[
  a < b, \quad a = b, \quad a > b.
  \]

- For all $a, b, c \in F$, if $a < b$, then $a + c < b + c$.
- For all $a, b \in F$, if $a > 0$ and $b > 0$, then $ab > 0$.

For example, $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, while $\mathbb{C}$ is not.

**Definition 4.** Suppose that $F$ is an ordered field and $S$ is a subset of $F$. An upper bound for $S$ is any element $M$ of $F$ so that $M \geq x$ for all $x \in S$. A least upper bound for $S$ is any element $L$ which is an upper bound for $S$ and which also has the property that every $a < L$ is not an upper bound for $S$: $L$ is the smallest upper bound for $S$.

An ordered field $F$ has the least upper bound property if any nonempty subset $S \subseteq F$ with an upper bound has a least upper bound.

For example, $\mathbb{Q}$ does not have the least upper bound property.

**Theorem 5.** There is an ordered field $\mathbb{R}$, the field of real numbers, which has the least upper bound property and contains $\mathbb{Q}$ as a subfield.

We will not prove this theorem. Instead, we will essentially use it as our definition of the field of real numbers.