Ordered fields (Taylor and Mann, §2.1–2.2)

Definition 1. A *field* is a set F with operations "addition" and "multiplication" satisfying the following:

- If a and b are in F, then so are $a+b$ and ab.
- Addition and multiplication are each associative and commutative, and together they are distributive: $a(b+c) = ab+ac$ for all $a,b,c \in F$.
- F contains elements called 0 and 1 satisfying:

  $$a+0=a \quad \text{and} \quad a \cdot 1 = a$$

  for all $a \in F$.
- For all $a \in F$, there exists an element $b \in F$ so that $a+b=0$.
- For all $a \in F$ with $a \neq 0$, there exists an element $b \in F$ so that $ab = 1$.

Example 2. The set of rational numbers $\mathbf{Q}$, the set of real numbers $\mathbf{R}$, and the set of complex numbers $\mathbf{C}$ each form fields. The integers $\mathbf{Z}$ and the non-negative integers $\mathbf{N}$ do not.

Definition 3. A field F is *ordered* if it has an ordering $<$ so that:

- For all $a,b \in F$, exactly one of these holds:

  $$a < b, \quad a = b, \quad a > b.$$
- For all $a,b,c \in F$, if $a < b$, then $a+c < b+c$.
- For all $a,b \in F$, if $a > 0$ and $b > 0$, then $ab > 0$.

For example, $\mathbf{Q}$ and $\mathbf{R}$ are ordered fields, while $\mathbf{C}$ is not.

Definition 4. Suppose that F is an ordered field and S is a subset of F. An *upper bound* for S is any element M of F so that $M \geq x$ for all $x \in S$. A *least upper bound* for S is any element L which is an upper bound for S and which also has the property that every $a < L$ is not an upper bound for S: L is the smallest upper bound for S.

An ordered field F has the *least upper bound property* if any nonempty subset $S \subseteq F$ with an upper bound has a least upper bound.

For example, $\mathbf{Q}$ does not have the least upper bound property.

Theorem 5. *There is an ordered field* $\mathbf{R}$*, the* field of real numbers*, which has the least upper bound property and contains* $\mathbf{Q}$ *as a subfield.*

We will not prove this theorem. Instead, we will essentially use it as our definition of the field of real numbers.