Mathematics 327 Midterm Exam Name: Answers

May 8, 2009

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. (10 points) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Just using the definition of convergence, prove that

$$\lim_{n\to\infty} (a_n + b_n) = A + B.$$

(Don't just cite a theorem – prove this straight from the definitions.)

Solution: Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there is an integer N so that for all $n \geq N$, we have $|A - a_n| < \varepsilon/2$. Since $\lim_{n\to\infty} b_n = B$, there is an integer N' so that for all $n \geq N'$, we have $|B - b_n| < \varepsilon/2$. Let $N'' = \max(N, N')$. Then for all $n \geq N''$, we have

$$\begin{aligned} |(A+B) - (a_n + b_n)| &= |(A - a_n) + (B - b_n)| \\ &\leq |A - a_n| + |B - b_n| \quad \text{(triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} (a_n + b_n) = A + B.$$

2. (a) (4 points) Prove that $2^n > n$ for every positive integer n.

   (b) (6 points) The Archimedean property of the real numbers says

   For any positive real numbers c and d, there is a positive integer n so that $nc > d$.

   Use part (a) and the Archimedean property to prove that if a and b are real numbers with $a < b$, then there are integers m and n with $n > 0$ so that

$$a < \frac{m}{2^n} < b.$$

Solution: (a) We prove this by induction: if $n = 1$, the inequality in question is $2^1 > 1$, which is true. Assume that $2^n > n$ for some $n \geq 1$. Then $2^{n+1} = 2 \cdot 2^n > 2n$ by the inductive hypothesis. Furthermore, $2n = n + n \geq n + 1$ (since $n \geq 1$). Combining the inequalities, we see that $2^{n+1} > n+1$. This finishes the inductive step, and hence the proof.

(b) Since $a < b$, the number $b - a$ is positive. Apply the Archimedean property to the numbers $c = b - a$ and $d = 1$: there is a positive integer n so that $n(b - a) > 1$. By the previous part (which applies since n is a positive integer), we also have $2^n > n$, so $2^n(b-a) > n(b-a) > 1$. Let m be the smallest integer so that $m > 2^n a$. Then $m-1 \leq 2^n a$, so

$$2^n a < m \leq 2^n a + 1 < 2^n a + 2^n(b - a) = 2^n b,$$

and so

$$a < \frac{m}{2^n} < b.$$

Alternatively, once we know that $2^n(b - a) > 1$, we rewrite this as $2^n b - 2^n a > 1$: the numbers $2^n b$ and $2^n a$ differ by more than 1. Therefore there is an integer m in between them: there is an integer m with $2^n b > m > 2^n a$. Divide by 2^n to get the result.

3. Let S be the set of all numbers of the form $(-1)^n - (1/n)$, $n = 1, 2, 3, \dots$. Answer the following, giving brief justifications for your answers.

   (a) (5 points) Find the least upper bound and greatest lower bound of S.

   Solution: First, the points of S are -2, $1/2$, $-4/3$, $3/4$, $-6/5$, $5/6$, ... The greatest lower bound is -2: this is a lower bound, and since it is in S, it must be the greatest lower bound (any larger number will be larger than -2, and so won't be a lower bound). The least upper bound is 1: the negative terms are certainly less than 1, and the positive terms are of the form $1 - 1/n$ with n even; these are also less than 1. Since the positive terms increase and approach 1, 1 is the least upper bound.

   (b) (5 points) Find all of the accumulation points of S.

   Solution: There are two accumulation points: 1 and -1. Since the even terms approach 1 and the odd terms approach -1, they are both accumulation points. To see that they are the only two, note that for any other point x on the real line, it is easy to find a neighborhood of x which does not contain infinitely many points of S, and therefore x cannot be an accumulation point of S.

   (c) (5 points) Is S open?

   Solution: No: the number -2 is in S, but no neighborhood of -2 is contained in S. (This same argument holds for any point of S, in fact.)

   (d) (5 points) Is S closed?

   Solution: No. There are at least two good reasons for this: the number 1 is an accumulation point of S but is not in S, so S does not contain all of its accumulation points, and so is not closed (by a theorem in the book). Alternatively, the complement S^c of S contains 1, but every neighborhood of 1 contains points of S, and hence no neighborhood of 1 is completely contained in S^c. Since S^c is not open, S is not closed.