Mathematics 327 Final Exam

June 8, 2009

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc. Provide reasons for all of your answers.

1. (10 points) For which integers $a$ does the following series converge? For which integers $a$ does it diverge?

\[ \sum_{n=1}^{\infty} \frac{n^n}{n!} a^n \]

Solution: Use the ratio test. Write the $n$th term as $u_n$; then the ratio $|u_{n+1}/u_n|$ is

\[ \frac{|u_{n+1}|}{u_n} = \frac{(n+1)^n |a|^{n+1}}{(n+1)!} \frac{n!}{n^n |a|^n} = \frac{(n+1)^n |a|}{n^n} = \left(1 + \frac{1}{n}\right)^n |a|. \]

As $n$ goes to infinity, this approaches $e|a|$. Therefore if $|a| < 1/e$, this ratio is less than 1 and the series converges. If $|a| > 1/e$, it diverges. More explicitly, the series converges if $a = 0$, and it diverges for all integers $a$ with $|a| \geq 1$.

2. (10 points) Abel's test says:

Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent, and that $b_n > 0$ and $b_n \geq b_{n+1}$ for all $n \geq 0$.

Then $\sum_{n=0}^{\infty} a_n b_n$ is convergent.

Prove this.

Solution: I’m going to apply Dirichlet’s test. The $a_n$’s satisfy the conditions of that test: the requirement is that the partial sums $\sum_{i=0}^{n} a_i$ be bounded, and since the series $\sum a_i$ converges, the sequence of partial sums converges, and hence they are bounded.

The $b_n$’s, though, don’t necessarily satisfy the condition that $b_n \to 0$. However, since the $b_n$’s are decreasing and bounded below, the limit $\lim b_n$ exists, so let $b = \lim b_n$ and for each $n$, let $c = b_n - b$. Then $c_n \geq 0$ and $c_n \geq c_{n+1}$ for all $n$, and also $c_n \to 0$ as $n \to \infty$. Therefore the $a_n$’s and the $c_n$’s satisfy the conditions for Dirichlet’s test, so the series $\sum a_n c_n$ converges. Plug in $c_n = b_n - b$; then we get

\[ \sum a_n b_n = \sum a_n c_n + \sum a_n b. \]

Each sum on the right side converges; therefore the one on the left does as well.

(Note that the comparison test requires that all of the terms in the series be non-negative. We don’t know anything about the signs of the series $\sum a_n$, so we can’t use a comparison test here, at least not in any simple way.)
3. (As in the text book, “log” means the natural log.)

(a) (5 points) Does the series \( \sum_{n=2}^{\infty} \frac{1}{n \log n} \) converge or diverge?

**Solution:** It converges. I’ll verify this with the integral test: the series converges if and only if the improper integral
\[
\int_{2}^{\infty} \frac{1}{x \log x} \, dx
\]
converges. Do this integral with a substitution: let \( u = \log x \), so \( du = dx/x \). Then the integral equals
\[
\int_{\log 2}^{\infty} \frac{du}{u^2} = \lim_{a \to \infty} -\frac{1}{u} \bigg|_{\log 2}^{a} = \frac{1}{\log 2}.
\]
Since the integral converges, so does the series.

(b) (5 points) Does the series \( \sum_{n=1}^{\infty} \frac{1}{n \log n} \) converge or diverge?

**Solution:** It converges. I’ll verify this using a comparison test. For \( n \geq 3 \), we have \( \log n \geq \log 3 > 1 \). Therefore for \( n \geq 3 \), we have \( \frac{1}{n \log n} < \frac{1}{n^{\log 3}} \). Since \( \log 3 \) is bigger than 1, the series \( \sum \frac{1}{n^{\log 3}} \) converges; therefore the original series converges as well.

(c) (5 points) For which real numbers \( x \) does the series \( \sum_{n=1}^{\infty} \frac{x^n}{3n} \) converge, and for which does it diverge?

**Solution:** Let’s use the root test: let \( u_n = \frac{x^n}{3n} \), and then
\[
\sqrt[n]{|u_n|} = \frac{|x|}{(3n)^{1/n}}.
\]
As \( n \to \infty \), \( n^{1/n} \) goes to 1, as does \( 3^{1/n} \); therefore \( \sqrt[n]{|u_n|} \) goes to \( |x| \). So if \( |x| < 1 \), the series converges and if \( |x| > 1 \), it diverges. What if \( |x| = 1 \)? If \( x = 1 \), we have a harmonic series, which diverges. If \( x = -1 \), we have a series which converges by the alternating series test. Summarizing: if \( -1 \leq x < 1 \), the series converges; otherwise, it diverges.

The ratio test works just as well.
4. (15 points) Define $f(x)$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

for all real numbers $x$.

(a) Find a simple expression for $f(x)$ when $x \neq 0$. (Hint: factor out $x^2$ and use a geometric series.)

**Solution:** Note that if $x \neq 0$, then $1 + x^2 > 1$, so $\frac{1}{1+x^2} < 1$. We have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n}$$

$$= x^2 \sum_{n=0}^{\infty} \left( \frac{1}{1+x^2} \right)^n = x^2 \frac{1}{1 - \frac{1}{1+x^2}} \quad \text{(geometric series)}$$

$$= x^2 \frac{1 + x^2}{1 + x^2 - 1} = x^2 \frac{1 + x^2}{x^2}$$

$$= 1 + x^2.$$

(b) What is $f(0)$? Does $f(x)$ have any discontinuities? Can you deduce anything about uniform convergence?

**Solution:** Clearly $f(0) = 0$: just plug $x = 0$ into the series defining $f(x)$. (The answer for part (a) explicitly says $x \neq 0$, so we can’t plug $x = 0$ into that to find $f(0)$.) Therefore $f(x)$ has a discontinuity at 0: when $x \neq 0$, $f(x) = 1 + x^2$, and as $x$ approaches zero, this approaches 1. Therefore, the series cannot converge uniformly on any interval containing 0.

(c) Show that if $a$ is any positive real number, then the series converges uniformly on the interval $[a, \infty)$.

**Solution:** Fix a positive number $a$. Then for all $x \in [a, \infty)$, since $x \geq a$, then $\frac{1}{1+x^2} \leq \frac{1}{1+a^2}$. Since $a$ is positive, $1/(1+a^2) < 1$, so the geometric series $\sum 1/(1+a^2)^n$ converges. By the Weierstrass $M$-test, the series defining $f(x)$ converges uniformly.
5. (10 points) Let \( f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \) for \( 0 \leq x \leq 2\pi \).

(a) Does the series converge uniformly to \( f(x) \)?

**Solution:** Yes: for all \( n \) and \( x \), we have
\[
\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2},
\]
and so by the Weierstrass \( M \)-test, the series converges uniformly.

(b) Is the equality
\[
\frac{df}{dx}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}
\]
valid for all \( x \) in \([0, 2\pi]\)? (Or as the book phrases it, can \( f'(x) \) be calculated for each \( x \) in the specified interval by differentiating the series for \( f(x) \) term by term?)

**Solution:** No. Consider the series
\[
\sum_{n=1}^{\infty} \frac{d}{dx} \left( \frac{\sin nx}{n^2} \right) = \sum_{n=1}^{\infty} \frac{n \cos nx}{n^2} = \sum_{n=1}^{\infty} \frac{\cos nx}{n}.
\]
When \( x = 0 \), this becomes \( \sum_{n=1}^{\infty} \frac{1}{n} \), which is the harmonic series, and hence diverges. So the series of derivatives doesn’t converge at all, let alone uniformly.