Axioms and properties of the real numbers

The basic axioms of addition and multiplication: Given two real numbers a and b, they have a sum $a + b$ and a product ab which are also real numbers. These satisfy the following properties, for all real numbers a, b, and c:

- (Well-defined) If $a = b$, then $a + c = b + c$ and $ac = bc$.
- (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- (Commutativity) $a + b = b + a$ and $ab = ba$.
- (Distributivity) $a(b + c) = ab + ac$.
- (Zero) There is a number 0 which satisfies $a + 0 = 0$.
- (One) There is a number 1 which satisfies $a \times 1 = a$.
- (Negatives) The equation $a + x = 0$ has the unique solution $x = -a$.
- (Reciprocals) If $a \neq 0$, the equation $ax = 1$ has the unique solution $x = 1/a = a^{-1}$.

Some consequences (for any real numbers a, b, c):

- (Cancellation) If $a + c = b + c$, then $a = b$.
- (Cancellation) If $c \neq 0$ and $ac = bc$, then $a = b$.
- $a \times 0 = 0 = 0 \times a$.
- $(-a)b = -(ab) = a(-b)$, $(-a)(-b) = ab$.

The basic axioms of inequalities: There is an ordering $<$ on real numbers, satisfying the following, for all real numbers a, b, and c:

- (Transitivity) If $a < b$ and $b < c$, then $a < c$.
- (Trichotomy) For any a and b, exactly one of the following is true: $a < b$, $a = b$, $a > b$.
- $a < b$ if and only if $a + c < b + c$.
- If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$.

Some consequences (for any real numbers a, b, c):

- $a > 0$ if and only if $-a < 0$.
- For all a, $a^2 \geq 0$. If $a \neq 0$, then $a^2 > 0$.
- In particular, $1 > 0$.
- If $c > 0$, then $a < b \Leftrightarrow ac < bc$.
- If $c < 0$, then $a < b \Leftrightarrow ac > bc$.
- If $a > 0$ and $b > 0$, then $ab > 0$.