**Topics**

Basic logic and mathematical language; implications. Proofs: direct, contradiction, induction.

Set theory and quantifiers. Functions; injections, surjections, bijections.

Counting. Finite sets, their properties and their cardinality. Infinite sets, their properties and their cardinality. Number systems: $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

The division theorem, greatest common divisors, the Euclidean algorithm, Bézout's theorem (Theorem 17.1.1).

Prime numbers, the Fundamental Theorem of Arithmetic. Modular arithmetic.

**Sample questions**

1. Write the negation of each of the following statements without using phrases of negation (such as "it is not true that . . . "). No justifications are required for this problem.

   (a) For all $a, x \in \mathbb{R}$, there is a unique $y$ such that $x^4y + ay + x = 0$.

   (b) For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|x-y| < \delta$ implies that $|f(x)-f(y)| < \varepsilon$.

   (c) There exist natural numbers $a$ and $b$ such that $\frac{a}{b} = \sqrt{2}$.

2. Use induction to prove the following properties of integer exponents: for all real numbers $x$ and $y$ and all non-negative integers $m$ and $n$,

   (a) $x^n y^n = (xy)^n$

   (b) $x^{m+n} = x^m x^n$

   (c) $(x^m)^n = x^{mn}$

   (Start with the basic properties of addition and multiplication given in 2.3.1 and the inductive definition of integer exponents in 5.3.3.)

3. Fix an integer $k \geq 2$. Prove that $k-1$ divides $k^n - 1$ for all integers $n \geq 1$.

4. Prove by induction on $n$ that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

   for all positive integers $n$.

5. Prove that $\sqrt{2}$ is not a rational number.

6. Let $A$ and $C$ be sets and let $f : A \to C$ and $g : C \to A$ be functions. True or false: if $f \circ g$ and $g \circ f$ are both bijective, then $f$ is bijective. If it's true, prove it. If it's false, give a counterexample.

7. For each of the following functions, tell me whether it is injective, and tell me whether it is surjective. (That is, for each of these, write "injective" or "not injective", and write "surjective" or "not surjective".) Give brief justifications for your answers.

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x^2}$

   (b) $g : \mathbb{Z}^{>0} \to [-1, 1]$ defined by $g(n) = \sin(2\pi n)$

   (c) $h : \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ defined by $h(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

8. Prove that $\mathbb{Q}$ is denumerable.

9. Suppose that $A$ and $B$ are denumerable sets. Prove that $A \times B$ is denumerable.

10. Suppose that for each positive integer $n$, $A_n$ is a denumerable set. Prove that the union

$$\bigcup_{n \geq 1} A_n = A_1 \cup A_2 \cup A_3 \cup \ldots$$

    is denumerable.

11. Describe the Euclidean algorithm when applied to integers $a$ and $b$, not both zero. Prove that it must terminate.

12. Suppose that $a$, $b$, and $c$ are integers. Use Bézout's theorem to prove that if $(a, c) = 1$ and $c|ab$, then $c|b$.

13. (exercise 23.5) *Euler's totient function* $\phi : \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ is defined by setting $\phi(n)$ equal to the number of positive integers less than or equal to $n$ which are coprime to $n$. Prove that if $p$ is prime and $k$ is a positive integer, then $\phi(p^k) = p^k - p^{k-1}$.

14. Problems VI: 4, 12

15. It's also a good idea to review the homework problems for the quarter.