Mathematics 134 Quiz 6

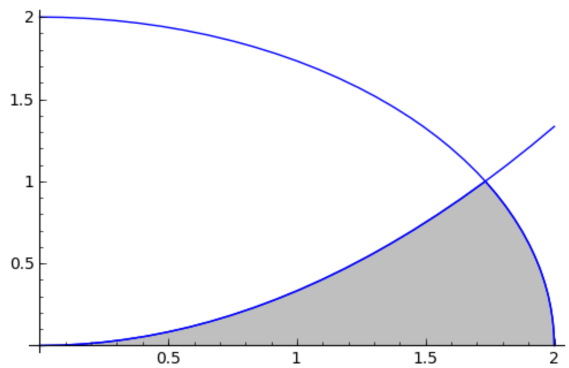
Name: Answers

November 12, 2009

Instructions: This is a closed book quiz, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. Consider the region in the first quadrant bounded by the x-axis, the curve $y = x^2/3$, and the curve $x^2 + y^2 = 4$. Represent its area by one or more integrals. *Do not evaluate the integral(s).*

Solution: The region in question is drawn here:



Note that the two curves intersect at the point $(\sqrt{3}, 1)$.

We can integrate with respect to x or with respect to y; either approach is valid and should earn full credit (if done correctly).

To integrate with respect to x, we divide up into two integrals, one to the left of the intersection point and one to the right. For the left piece, we are finding the area under the parabola $y = x^2/3$, and for the right piece, we are finding the area under the arc of the circle $y = \sqrt{4 - x^2}$:

$$\boxed{\int_0^{\sqrt{3}} \frac{x^2}{3}\, dx + \int_{\sqrt{3}}^2 \sqrt{4 - x^2}\, dx}$$

If we integrate with respect to y, we just have a single integral, but we need to solve for x in terms of y and compute the area between the parabola ($x = \sqrt{3y}$) and the circle ($x = \sqrt{4 - y^2}$). We are integrating as y goes from 0 to 1. The circle is the larger function, so the integral is

$$\boxed{\int_0^1 \left(\sqrt{4 - y^2} - \sqrt{3y}\right) dx}$$

2. A rod of length L is placed on the x-axis from $x = 0$ to $x = L$. The density of the rod varies directly as the distance from the $x = 0$ endpoint of the rod; that is, the density is proportional to x. Find the point $x = a$ that divides the rod into two pieces of equal mass.

Solution: Write the density function as $\lambda(x) = kx$.

If we cut the rod at $x = a$, then there are two pieces, one from $x = 0$ to $x = a$, and one from $x = a$ to $x = L$. So we equate their masses and solve for a:

$$\int_0^a kx\,dx = \int_a^L kx\,dx.$$

The k's cancel, and once we integrate we get factors of $1/2$ on each side which also cancel. We end up with

$$a^2 = L^2 - a^2, \quad \text{or} \quad 2a^2 = L^2.$$

Since a must be positive because of the setup of the problem, we see that $a = L/\sqrt{2}$.