Mathematics 134 Quiz 4 Name: Answers

October 22, 2009

Instructions: This is a closed book quiz, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. Show that the equation $x^3 + 9x^2 + 33x - 8 = 0$ has exactly one real root.

Solution: Let $p(x) = x^3 + 9x^2 + 33x - 8$. Since $p(x)$ is an odd-degree polynomial, it has at least one real root (by the Intermediate Value Theorem). If it had more than one, say $p(a) = 0$ and $p(b) = 0$, then by Rolle's theorem, there would be a point c between a and b with $p'(c) = 0$. In this case, $p'(x) = 3x^2 + 18x + 33 = 3(x^2 + 6x + 11)$. This has no real roots: $p'(c) = 0$ never holds. Therefore $p(x)$ itself cannot have more than one real root.

2. Let $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$. Under what conditions on a, b, and c will $f(x)$ have exactly one critical point? In this situation, can you tell whether that critical point is a local max, a local min, or neither?

Solution: The critical points are the points where $f'(x) = 0$ or $f'(x)$ does not exist. For this function, the derivative always exists, so we focus on the first case. We see that $f'(x) = 3ax^2 + 2bx + c$, and this is zero (according to the quadratic formula) at points

$$\frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{6a}.$$

So there is exactly one such point if $b^2 - 3ac = 0$.

I claim that it is neither a max nor a min, by the first derivative test: $f'(x)$ is a quadratic with a single zero, so we know from its graph that, except for its root, it is always positive (if $3a > 0$) or always negative (if $3a < 0$). Therefore the first derivative does not change signs at the critical point, so the critical point is not a max or a min.