Mathematics 134 Quiz 3

Name: Answers

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Instructions: This is a closed book quiz, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. Define a function $g(x)$ by

$$g(x) = \begin{cases} x^2, & \text{if } x \text{ rational,} \\ 0, & \text{if } x \text{ irrational.} \end{cases}$$

Show that $g(x)$ is differentiable at $x = 0$ and compute $g'(0)$.

Solution: We examine the limit

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h)}{h}.$$

When h is rational, $g(h)/h = h^2/h = h$. When h is irrational, $g(h)/h = 0$. So as h approaches 0, $g(h)/h$ also approaches 0. (To prove this carefully, we can use the ε's and δ's, or we can use the pinch theorem: the function $g(h)/h$ lies between the functions 0 and h, and so as $h \to 0$, $g(h)/h \to 0$.) Therefore the limit above exists and equals zero, so $g(x)$ is differentiable at $x = 0$ with derivative $g'(0) = 0$.

2. Find the equations for all lines which pass through the point $(0, 2)$ and are tangent to the curve $y = x^3$.

Solution: The equation for the line tangent to the curve $y = x^3$ through the point (a, a^3) is given by the equation

$$y - a^3 = (3a^2)(x - a) \quad \text{(because } 3a^2 \text{ is the derivative at that point)}$$

or

$$y = 3a^2x - 2a^3.$$

If this passes through the point $(x, y) = (0, 2)$, then (plugging in $x = 0$ and $y = 2$) we have $2 = -2a^3$, so $a^3 = -1$, so $a = -1$. Thus there is only one such line, the tangent line at the point $(-1, -1)$, with equation

$$y = 3x + 2.$$