Topics for Midterm #1

Chapter 1: induction

Chapter 2: Definition of limit; basic limit theorems, limits of polynomials; pinch theorem, limits of trig functions; continuity; intermediate value theorem, extreme value theorem

Chapter 3: Definition of derivative; basic derivative formulas, derivatives of polynomials and trig functions; chain rule; implicit differentiation, derivative of $x^{p/q}$

Chapter 4: Mean value theorem: proof, Rolle's theorem; increasing and decreasing functions; local extreme values, extreme values when working over a closed interval; concavity, inflection points; curve sketching; max-min problems

Sample questions for Midterm #1

1. Give the definition of $\lim_{x \to a} f(x)$.

2. Explain why the equation $6x^4 - 7x + 1 = 0$ has at most two real roots.

3. Let $f(x) = x^2$. Using only the definition of limit, prove that $\lim_{x \to 0} f(x) = 0$.

4. Let $f(x) = x^2 + x + 1$. Using only the definition of continuity (and limit theorems), prove that f is continuous at the origin.

5. Let $P(x) = x^3 + x + 1$. At which, if any, points on the graph of P does the tangent line pass through the origin?

6. Let $f(x) = \sqrt{x^2 + 9}$. Compute $f'(x)$ directly from the definition of derivative as a limit of difference quotients.

7. Compute the indicated derivatives.

   (a) $\left.\frac{dy}{dx}\right|_{x=\pi, y=1}$, where $\tan(xy^2) = y + \sin 4x$.

   (b) $\frac{d^2 g(\cos 3x)}{dx^2}$ in terms of g' and g'', where $g : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function.

8. Let $f(x) = \frac{9}{x} - \frac{9}{x^2}$. Find the critical points and inflection points of f, and draw a (reasonably careful) sketch of its graph.

9. Show that the equation $x^6 + 6x^3 = 3$ has exactly two real solutions.

10. Suppose that f is continuous and $f(0) = 1$. Show, from the definition of continuity, that there is an interval I centered at 0 such that $f(x) > \frac{2}{3}$ for $x \in I$. (*Don't make this complicated; it's not.*)

11. Find the maximum possible area and the dimensions of a rectangle with sides parallel to the axes if its vertices lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \; a > 0 \text{ and } b > 0 \text{ fixed.}$$

12. Notice that the point $(x, y) = (0, 1)$ satisfies the equation

$$x^2 + \sin^2(x) + y^2 = 1\,.$$

One can show that this equation defines a twice differentiable function $y = f(x)$ for which $f(0) = 1$.

(a) Use implicit differentiation to compute $f'(0)$.

(b) Use implicit differentiation to compute $f''(0)$.

(c) Using the results of parts (a) and (b), characterize the point $(0, 1)$. (For instance, it might be a local maximum for f, a local minimum, a point of inflection, or something else.) Explain your answer.

13. (a) State the Mean-Value Theorem.

(b) Suppose that the function f is continuous for all $x \in \mathbb{R}$ and that $f(0) = 0$. Suppose further that f is differentiable for $x \neq 0$ and that f' satisfies the inequality

$$|f'(x)| \le |x|\,, \text{ for all } x \neq 0.$$

Use the Mean-Value Theorem to prove that f satisfies the inequality

$$|f(x)| \le x^2\,, \text{ for all } x.$$

Be sure to justify your use of the Mean-Value Theorem by verifying that all at hypotheses of the theorem are satisfied.

(c) Use the result of part (b) and the definition of the derivative as the limit of difference quotients to prove that f is differentiable at $x = 0$ and $f'(0) = 0$.

14. Consider the function $f : \mathbb{R} \to \mathbb{R}$. Suppose that f satisfies the inequality

$$|f(x) - f(y)| \le |x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Prove that f is continuous on $\mathbb{R}$

(b) Give an example to show that f need not be differentiable on all of $\mathbb{R}$.