8.3.1 Let $u = f(x,y)$ and $v = g(x,y)$. Then we have

$$u^3 + xv - y = 0, \quad v^3 + yu - x = 0,$$

and we want to solve for u and v near the point $(x_0, y_0, u_0, v_0) = (0,1,1,-1)$. We can do this if the Jacobian is nonzero. The Jacobian is

$$\det \begin{bmatrix} 3u^2 & x \\ y & 3v^2 \end{bmatrix} = 9u^2v^2 - xy.$$

At the point in question, this equals 9. Since it is nonzero, we can solve for u and v in terms of x and y. That is, we can find functions f and g as specified. The implicit theorem guarantees that they will be continuous in a neighborhood of $(0,1)$.

8.3.2 (a) The condition is that the Jacobian at the point is nonzero. The Jacobian, here, is

$$\det \begin{bmatrix} 2u & 2v & 2w \\ 2u & 2v & 0 \\ 2u & 0 & 2w \end{bmatrix} = -8uvw.$$

This is nonzero whenever u, v, and w are nonzero; therefore we can solve for u, v, and w in neighborhoods of all points $(x_0, y_0, z_0, u_0, v_0, w_0)$, as long as $u_0 \neq 0$, $v_0 \neq 0$, and $w_0 \neq 0$.

(b) Solving for u^2, v^2, and w^2 yields

$$\begin{aligned} u^2 &= -x^2 + y^2 + z^2, \\ v^2 &= x^2 - z^2, \\ w^2 &= x^2 - y^2. \end{aligned}$$

Suppose, for example, that we are working in a neighborhood of a point with $v_0 = 0$. If $v_0 = 0$, then $x_0^2 - z_0^2 = 0$, so $x_0 = \pm z_0$. At nearby points, x^2 will be close to z^2; in particular, at some nearby points, we will have $x^2 - z^2 < 0$, and at some nearby points we will have $x^2 - z^2 > 0$. If $x^2 - z^2 < 0$, then there is no solution: v^2 cannot be negative. This is bad. If $x^2 - z^2 > 0$, then there are two nearby solutions: $v = \pm\sqrt{x^2 - z^2}$. Having two solutions is also bad. (To solve for u, v, and w, we need to know that for each (x,y,z), there is a unique point (u,v,w), and here we either have no such point – if $x^2 - z^2 < 0$ – or more than one such point – if $x^2 - z^2 > 0$.)

Similar comments apply if $u_0 = 0$ or if $w_0 = 0$.

8.3.5 Let $F(x,y,z) = x^2 + y^2 + z^2 - r^2$, and let $G(x,y,z) = x + y + z - c$. According to the implicit function theorem, we need to know that the Jacobian

$$J = \det \begin{bmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{bmatrix} = \det \begin{bmatrix} 2y & 2z \\ 1 & 1 \end{bmatrix} = 2y - 2z$$

is nonzero at the point (x_0, y_0, z_0). Thus if we require that $y_0 \neq z_0$, then we will be able to solve for y and z in terms of x.

extra problem 1. Let $f(x,y) = y - \sin x$. We want to solve $f(x,y) = 0$ for x; thus we need to know that $\partial f/\partial x \neq 0$, which is the same as $-\cos x = 0$. Thus in neighborhoods of any point *except for* those of the form $(\frac{\pi}{2} + n\pi, \sin(\frac{\pi}{2} + n\pi))$, where n is an integer, we can solve for x in terms of y.

(You can also write these points as $(\frac{\pi}{2} + 2n\pi, 1)$ and $(\frac{3\pi}{2} + 2n\pi, -1)$, where n is an integer.)

2. Let $g(x,y) = y - \tan x$. We need to know that $\partial g/\partial x \neq 0$, which is to say, that $-\sec^2 x \neq 0$. Note that the domain of the tangent function is all real numbers except for those where $\cos x = 0$, and so for all points in the domain, $-\sec^2 x$ is nonzero, and so we can solve for x in terms of y. So the answer is "everywhere", if you are careful enough to interpret that to mean "everywhere in the domain of the tangent function"; alternatively, the answer is "all points of the form $(x, \tan x)$ where x is not equal to $\frac{\pi}{2} + n\pi$ for n an integer".