7.6.2 Let $F(x,y) = (a/x) + (b/y) + xy$. Note that x and y must be nonzero for this to be defined. Let's compute some partial derivatives:

$$F_1(x,y) = -\frac{a}{x^2} + y, \quad F_2(x,y) = -\frac{b}{y^2} + x,$$
$$F_{11}(x,y) = \frac{2a}{x^3}, \quad F_{12}(x,y) = 1, \quad F_{22}(x,y) = \frac{2b}{y^3}.$$

The critical points occur where $F_1(x,y) = 0 = F_2(x,y)$. Thus $y = a/x^2$ and $x = b/y^2$. Solving for x (and using the fact that $x \neq 0$) gives

$$x_0 = \left(\frac{a^2}{b}\right)^{1/3}, \quad y_0 = \left(\frac{b^2}{a}\right)^{1/3}.$$

We plug these into F_{11} and $F_{11}F_{22} - F_{12}^2$: at the point (x_0, y_0) we have

$$F_{11}(x_0,y_0) = \frac{2b}{a}, \quad F_{11}F_{22} - F_{12}^2 = \frac{2b}{a}\frac{2a}{b} - 1 = 3.$$

Thus $F_{11}F_{22} - F_{12}^2$ is always positive. If a and b have the same sign, then F_{11} is positive, and we get a minimum. If a and b have different signs, then F_{11} is negative, and we get a maximum.

7.6.3 Let $F(x,y) = (ax^2 + by^2)e^{-x^2-y^2}$. Then

$$F_1(x,y) = e^{-x^2-y^2}(2ax - (ax^2+by^2)2x), \quad F_2(x,y) = e^{-x^2-y^2}(2by - (ax^2+by^2)2y),$$

or

$$F_1(x,y) = e^{-x^2-y^2}2x(a - ax^2 - by^2), \quad F_2(x,y) = e^{-x^2-y^2}2y(b - ax^2 - by^2).$$

The first of these is zero when $x = 0$ or when $a = ax^2 + by^2$. The second is zero when $y = 0$ or $b = ax^2 + by^2$. Note that since $a \neq b$, we can't have both $a = ax^2 + by^2$ and $b = ax^2 + by^2$. If $x = 0$, the possibilities are $y = 0$, $y = -1$, $y = 1$. If $y = 0$, the possibilities are $x = 0$, $x = -1$, $x = 1$. So there are five critical points: $(0,0)$, $(0,-1)$, $(0,1)$, $(-1,0)$, and $(1,0)$.

To determine their "natures," we use the second derivative test, so we need to compute the various second partials. So:

$$F_{11}(x,y) = e^{-x^2-y^2}(-2ax^2 + 4ax^4 + 4bx^2y^2 + 2a - 8ax^2 - 2by^2),$$
$$F_{12}(x,y) = e^{-x^2-y^2}(-4axy - 4bxy + 4ax^3y - 4bxy^3),$$
$$F_{22}(x,y) = e^{-x^2-y^2}(-2by^2 + 4by^4 + 4ax^2y^2 + 2b - 8by^2 - 2ax^2).$$

Now we plug the critical points into F_{11} and $F_{11}F_{22} - F_{12}^2$. Note that if either $x = 0$ or $y = 0$ (as is the case for all of the critical points), then $F_{12} = 0$.

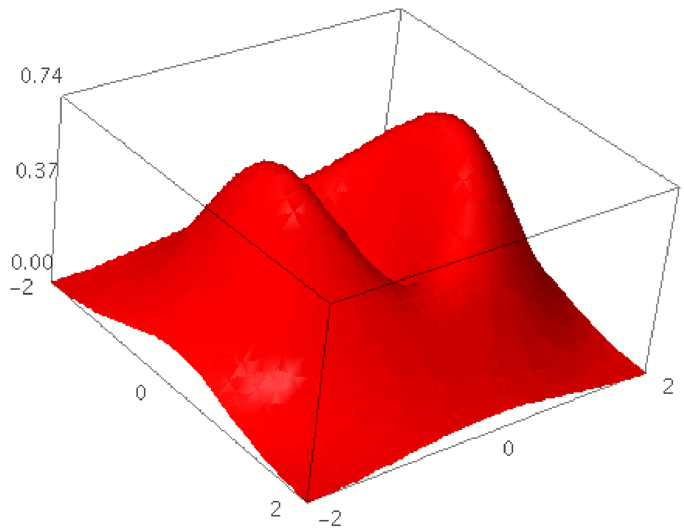
At $(0,0)$, $F_{11} = 2a$, $F_{12} = 0$, and $F_{22} = 2b$. Thus $F_{11}F_{22} - F_{12}^2 = 4ab > 0$. Thus this point is a minimum.

At $(0,-1)$, $F_{11} = e^{-1}(2a-2b) < 0$, $F_{12} = 0$, and $F_{22} = e^{-1}(-4b)$. Thus $F_{11}F_{22} - F_{12}^2 = e^{-2}(-4b)(2a-2b) > 0$, so this is a maximum.

The behavior at the point $(0,1)$ is just like that at $(0,-1)$: a maximum.

At $(-1,0)$ or $(1,0)$, $F_{11} = e^{-1}(-4a)$, $F_{12} = 0$, and $F_{22} = e^{-1})(2b-2a)$. Thus $F_{11} < 0$ and $F_{11}F_{22} - F_{12}^2 = e^{-2}(-4a)(2b-2a) < 0$. These points are saddles.

Here's a graph, in which you can see the two maxima, one of the two saddle points, and the suggestion of a minimum:



8.2.3 Let $F(x,y,z) = (x^2+y^2+z^2)^{1/2} - \cos z$. Then $F_2 = y(x^2+y^2+z^2)^{-1/2}$, and at the point $(0,1,0)$, this equals 1, so it can be solved for y in terms of x and z near this point.

On the other hand, $F_3 = z(x^2+y^2+z^2)^{-1/2} + \sin z$, and at the point $(0,1,0)$, this is equal to 0. Therefore the implicit function theorem does not apply, and we cannot conclude that it can be solved uniquely for z near this point.

8.2.5 Let $F(x,y,z) = z^3 + (x^2+y^2)z + 1$. Then $F_3 = 3z^2 + x^2 + y^2$. This is positive everywhere except at the point $(0,0,0)$. This point is not on the surface $F(x,y,z) = 0$. Therefore F_3 is nonzero at all points of the surface, so we can solve for z everywhere. Its partial derivatives are given by the usual formulas: for example,

$$\frac{\partial z}{\partial x} = -\frac{F_1}{F_3} = -\frac{2xz}{3z^2+x^2+y^2}.$$

This is continuous as long as the denominator is nonzero, and since it is nonzero at all points of the surface, this is continuous everywhere. A similar computation works for $\partial z/\partial y$.

Alternatively, you can view the equation $F(x,y,z) = 0$ as a cubic equation in z, and you can look up the solution to a cubic:

$$z = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(x^2+y^2)^3}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{(x^2+y^2)^3}{27}}}.$$

Computing the partials is more annoying, but at least they won't be in terms of z.

8.2.20 Let $F(x,y) = xe^y - y + 1$. To solve $F(x,y) = 0$ for y in terms of x near $(e^{-2}, 2)$, we need to know whether $\partial F / \partial y$ is nonzero at that point.

$$\frac{\partial F}{\partial y} = xe^y - 1.$$

This is zero at $(e^{-2}, 2)$, so we don't know if it is possible to write it in the form $y = f(x)$.

On the other hand,

$$\frac{\partial F}{\partial x} = e^y,$$

and this is nonzero everywhere, in particular at the point $(e^{-2}, 2)$. So we can solve for x in terms of y. Of course this is easy to do explicitly:

$$x = e^{-y}(y - 1).$$

From the plot below you can see that in fact it is not possible to solve for y in terms of x near the point $(e^{-2}, 2)$: to the left of that point, vertical lines hit the graph in two places, not one.