7.2.1 On the x-axis, the function $f(x,y)$ is equal to zero; thus $f_1(0,0)=0$. Similarly, $f_2(0,0)=0$. Away from the origin, we compute $f_1(x,y)$ and $f_2(x,y)$ by using the usual derivative formulas:

$$f_1(x,y) = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{\text{(big mess)}}{(x^2+y^2)^2},$$
$$f_2(x,y) = x\frac{x^2-y^2}{x^2+y^2} + xy\frac{\text{(big mess)}}{(x^2+y^2)^2}.$$

Thus at points $(0,y)$ with $y \neq 0$, we have

$$f_1(0,y) = y\frac{0-y^2}{0+y^2} = -y.$$

Since $f_1(0,0)=0$, we have $f_1(0,y)=-y$ for all y.

Similarly, at points $(x,0)$ with $x \neq 0$, we have

$$f_2(x,0) = x\frac{x^2}{x^2} = x,$$

and since $f_2(0,0)=0$, we have $f_2(x,0)=x$ for all x.

In general, to compute the partial derivative $g_2(a,b)$ of some function $g(x,y)$ at the point (a,b), you only need to know $g(a,y)$: fix the first coordinate and let the second coordinate vary. Thus you obtain $g_2(a,b)$ by taking the derivative of the function $g(a,y)$ with respect to y, and then plugging in $y=b$.

So in our situation, to compute $f_{12}(0,0)$, we take the derivative of $f_1(0,y)$ with respect to y, and then we plug in $y=0$. Similar remarks hold for $f_{21}(0,0)$. From the earlier calculations, therefore, we get

$$f_{12}(0,0) = -1, \quad f_{21}(0,0) = 1.$$

7.4.3 Let $(a,b)=(0,0)$ and let $(h,k)=(\pi/3,\pi/6)$. Then the mean value theorem says that

$$F\left(\frac{\pi}{3},\frac{\pi}{6}\right) - F(0,0) = F_1\left(\frac{\pi}{3}\theta,\frac{\pi}{6}\theta\right)\frac{\pi}{3} + F_2\left(\frac{\pi}{3}\theta,\frac{\pi}{6}\theta\right)\frac{\pi}{6}.$$

Computing F_1 and F_2, and then plugging in all of the numbers, yields the desired result.

7.5.1 Since all of $F(x,y)$, $F_1(x,y)$, $F_2(x,y)$, $F_{11}(x,y)$, and $F_{22}(x,y)$ have a factor of $\sin x$ or $\sin y$, they all give zero at the point $(x,y)=(0,0)$. On the other hand, $F_{12}(x,y)=\cos x\cos y$, so $F_{12}(0,0)=1$. We also note (for the reminder term) that $F_{111}(x,y) = -\cos x \sin y$, $F_{112}(x,y) = -\sin x \cos y$, $F_{122}(x,y) = -\cos x \sin y$, and $F_{222}(x,y) = -\sin x \cos y$. Thus Taylor's formula with $n=2$ gives

$$F(h,k) = hk + \frac{1}{3!}\left(-\cos(\theta h)\sin(\theta k)h^3 - 3\sin(\theta h)\cos(\theta k)h^2k - 3\cos(\theta h)\sin(\theta k)hk^2 - \sin(\theta h)\cos(\theta k)k^3\right),$$

for some θ between 0 and 1.

7.5.2 This is another straightforward computation. The answer is

$$F(h,k) = 1 - h^2 - k^2 + \frac{1}{3!}\left(\sin(\theta h)\cos(\theta k)h^3 + 3\cos(\theta h)\sin(\theta k)h^2k + 3\sin(\theta h)\cos(\theta k)hk^2 + \cos(\theta h)\sin(\theta k)k^3\right),$$

for some θ between 0 and 1.