Chain rule problem

(a) Plug $\vec{g}(t)$ into $f(x,y)$ to get $h(t)$:

$$h(t) = \begin{Bmatrix} \frac{(2t)(t^2)}{4t^2+t^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{Bmatrix} = \frac{2}{5}t.$$

Thus $h'(t) = 2/5$ for all t. In particular, $h'(0) = 2/5$.

(b) Write the components of $\vec{g}(t)$ as $(g_1(t), g_2(t))$. According to the chain rule,

$$h'(t) = \frac{\partial f}{\partial x}\frac{dg_1}{dt} + \frac{\partial f}{\partial y}\frac{dg_2}{dt}.$$

At $t = 0$, we have $\vec{g}(t) = (0,0)$, so this equation becomes

$$h'(0) = \frac{\partial f}{\partial x}(0,0)\frac{dg_1}{dt}(0) + \frac{\partial f}{\partial y}(0,0)\frac{dg_2}{dt}(0).$$

By computations like ones we've done several times before, we find that

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

Thus according to the chain rule, $h'(0) = 0$.

(c) Since the chain rule doesn't work here, some of its hypotheses must not hold for the functions $f(t)$ and $\vec{g}(t)$. The function $\vec{g}(t)$ is perfectly well-behaved, so we conclude that $f(t)$ is not differentiable at $(0,0)$.

(This is not too surprising, given its definition.)

6.6.2 We have $g(x) = F(x, f(x))$ for some unspecified function $F(x,y)$, and we know that $G(x, f(x)) = 0$. Let's try to compute $g'(x)$: by the chain rule,

$$g'(x) = \frac{d}{dx}F(x, f(x)) = F_1\frac{dx}{dx} + F_2\frac{df}{dx} = F_1 + F_2 f'(x).$$

We don't know what $f'(x)$ is, but we can extract some information about it from the condition $G(x, f(x)) = 0$. Let $h(x) = G(x, f(x))$. Then $h'(x) = 0$ (since $h(x) = 0$ for all x), but we can also compute $h'(x)$ using the chain rule:

$$0 = h'(x) = G_1 + G_2\frac{df}{dx}.$$

Therefore $f'(x) = -G_1/G_2$. (Indeed, this fits into the pattern in equation (6.6-4) in the book.) Therefore

$$g'(x) = F_1 + F_2(-G_1/G_2).$$

Put everything over a common denominator to get the desired result,

$$g'(x) = \frac{F_1G_2 - F_2G_1}{G_2}.$$

6.6.6 I will start by computing the various partial derivatives: by formula (6.6-4),

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}, \quad \left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}, \quad \left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

Multiplying these together yields

$$\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}\right)\left(-\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}\right)\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right) = (-1)^3 = -1.$$

6.6.11 Let $G(x,y,z) = F(x+y+z, x^2+y^2+z^2)$. Then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial z}} = -\frac{F_1 + 2xF_2}{F_1 + 2zF_2}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial G}{\partial y}}{\frac{\partial G}{\partial z}} = -\frac{F_1 + 2yF_2}{F_1 + 2zF_2}.$$

Now look at the quantity on the left side of the purported equation:

$$\begin{aligned}(y-x) + (y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} &= (y-x) - (y-z)\frac{F_1+2xF_2}{F_1+2zF_2} - (z-x)\frac{F_1+2yF_2}{F_1+2zF_2} \\ &= (y-x)\frac{F_1+2zF_2}{F_1+2zF_2} - (y-z)\frac{F_1+2xF_2}{F_1+2zF_2} - (z-x)\frac{F_1+2yF_2}{F_1+2zF_2} \\ &= \frac{(y-x)(F_1+2zF_2) + (z-y)(F_1+2xF_2) + (x-z)(F_1+2yF_2)}{F_1+2zF_2}.\end{aligned}$$

Now everything in the numerator cancels, and you get zero.

6.8.1 We want to maximize the volume $V = xyz$ of a box with side lengths x, y, and z, subject to the constraint that the corner (x,y,z) lies on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where $a,b,c > 0$ and $x,y,z \geq 0$.

Following Lagrange's method, we set

$$u = xyz + \lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)$$

and set the partial derivatives of u equal to zero:

$$yz + \lambda/a = 0, \quad xz + \lambda/b = 0, \quad xy + \lambda/c = 0.$$

Now do some algebra. For example, you might use the first equation to write λ in terms of y and z, plug that into the second and third equations to write each of x and z in terms of y. Then plug all of these into the restraint equation and solve for y. Once you know y, you can get x and z. The result is: there is only one critical point, and it occurs when $x = a/3$, $y = b/3$, $z = c/3$. The corresponding volume is $V = abc/27$.

Why is this a maximum? Well, we should check the boundary points; these occur when $x = 0$, $y = 0$, or $z = 0$. Clearly the volume is zero in all these cases. Since we are working with a closed region (the portion of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ for which $x \geq 0$, $y \geq 0$, and $z \geq 0$), the function is guaranteed to have a maximum value. Since it is positive away from the border, that maximum value does not occur on the border, so it occurs at a critical point. Since there is only one critical point, that must be the maximum. (In contrast, the minimum value is zero, attained all along the boundary.)

6.8.3 Suppose that the triangle in question has angles x, y, and z; then we have the constraint that $x+y+z = \pi$ (and that $x,y,z > 0$). We want to show that the function $F(x,y,z) = \sin(x)\sin(y)\sin(z)$, subject to this constraint, attains a maximum when $x = y = z$. We set

$$u = \sin(x)\sin(y)\sin(z) + \lambda(x+y+z)$$

and set the partial derivatives of y equal to zero:

$$\cos x \sin y \sin z + \lambda = 0, \quad \sin x \cos y \sin z + \lambda = 0, \quad \sin x \sin y \cos z + \lambda = 0.$$

Subtract the second equation from the first and do a little cancellation to get

$$\cos x \sin y = \sin x \cos y,$$

and thus $\cot x = \cot y$. (You could use tangent instead of cotangent, but $\tan x$ will be undefined if $x = \pi/2$, while $\cot x$ is defined for all x with $0 < x < \pi$.) Similarly, you can get $\cot y = \cot z$. The cotangent function is one-to-one on angles between 0 and π, so this means that $x = y = z$. Therefore the only critical point occurs when the triangle is equilateral and all of the angles are $\pi/3$. Since $\sin \pi/3 = \sqrt{3}/2$, the function has the value $3\sqrt{3}/2$ at this point.

To see that this critical point gives a maximum, we allow x, y, and z to be 0, and then we are working over a closed region: the portion of the plane $x+y+z = \pi$ where $0 \leq x,y,z \leq \pi$. Along the boundary, at least one of x, y, z will be zero, in which case its sine will be zero, and so the function $F(x,y,z)$ will be zero. This is less than the value at the critical point, so the critical point must give the maximum.