6.3.2 Let $f(x,y) = xy(c-x-y)$. Since we are working over a closed region, we know that $f(x,y)$ achieves its maximum somewhere either at a boundary point or at a critical point in the interior. We can see that $f(x,y) = 0$ for points (x,y) on the boundary, because for those points either $x = 0$, $y = 0$, or $x+y = c$ (so $c-x-y = 0$). In contrast, $f(x,y)$ is positive for points in the interior of the triangle (because in the interior, $x > 0$, $y > 0$, and $c > x+y$ so $c-x-y > 0$, and thus $f(x,y)$ is the product of three positive factors), so the maximum value will be at a critical point in the interior. We look for the critical points by computing the partial derivatives

$$\frac{\partial f}{\partial x} = cy - 2xy - y^2 = y(c-2x-y),$$
$$\frac{\partial f}{\partial y} = cx - 2xy - x^2 = x(c-x-2y)$$

and setting them equal to zero:

$$y(c-2x-y) = 0, \quad x(c-x-2y) = 0.$$

If a product $y(c-2x-y)$ is zero, then one of the factors must be zero. We may exclude the case $y = 0$, since that would give us a point on the boundary, and similarly for the other equation. Thus we have the equations

$$c-2x-y = 0, \quad c-x-2y = 0.$$

Solve these for x and y to get

$$x = c/3, \quad y = c/3.$$

This is the only critical point, and as argued above, this means that this must be where $f(x,y)$ achieves its maximum. The maximum value is therefore

$$f(c/3, c/3) = c^3/27.$$

6.3.4 Note that as x goes to infinity, x^2 goes to infinity. Since $f(x,y) = x^2 + \text{(other terms)}$ and each of the other terms is positive, $f(x,y)$ goes to infinity as x goes to infinity. Therefore $f(x,y)$ has no absolute maximum.

(Indeed, as $x^2+y^2 \to \infty$, that is, as (x,y) moves in any direction in the first quadrant radially away from the origin, $f(x,y)$ goes to ∞. Because of the terms $(576/x)$ and $(576/y)$, $f(x,y)$ also goes to infinity as you approach the x-axis or the y-axis.)

6.3.6 First we find the critical points by setting the partial derivatives equal to zero:

$$2y - x(1-x^2-y^2)^{-1/2} = 0,$$
$$2x - y(1-x^2-y^2)^{-1/2} = 0.$$

The first one says that $x = 2y(1-x^2-y^2)^{1/2}$. Plug this into the second one:

$$4y(1-x^2-y^2)^{1/2} - y(1-x^2-y^2)^{-1/2} = 0,$$

or

$$y(4(1-x^2-y^2)-1) = 0.$$

Therefore either $y = 0$ or $1-x^2-y^2 = 1/4$. In the first case we find that $x = 0$, and we have a critical point $(0,0)$. In the second case we get $(1-x^2-y^2)^{1/2} = 1/2$, so our equations become

$$2y-2x = 0, \quad 2x-2y = 0.$$

Thus we have $x = y$ and $1-x^2-y^2 = 1/4$, so we get two critical points: $(-\sqrt{3/8}, -\sqrt{3/8})$ and $(\sqrt{3/8}, \sqrt{3/8})$.

We plug the three critical points into the function:

$$f(0,0) = 1, \quad f(-\sqrt{3/8}, -\sqrt{3/8}) = 5/4, \quad f(\sqrt{3/8}, \sqrt{3/8}) = 5/4.$$

Now we look at the boundary points: we assume that $x^2+y^2=1$. For these points, $f(x,y)=2xy$. We can do this directly (by substituting $y=\pm\sqrt{1-x^2}$ and finding the max/min for the resulting two functions $g(x)=x\sqrt{1-x^2}$ and $h(x)=-x\sqrt{1-x^2}$), but it also might be fun to switch to polar coordinates: $x=r\cos\theta=\cos\theta$ (since $r=1$) and $y=\sin\theta$. Thus we want to find the max/min for the function $F(\theta)=2\cos\theta\sin\theta$ for $0\le\theta\le 2\pi$. The function F is zero at the end points; let's look for critical points.

$$F'(\theta)=-2\sin^2\theta+2\cos^2\theta=2\cos(2\theta).$$

This is zero when $\theta=\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. Using the second derivative test, or just plugging in these points, shows that $F(\theta)$ has a maximum value of 1 and a minimum value of -1. (You also might be able to recognize that $F(\theta)=\sin 2\theta$, a function which oscillates between -1 and 1, so you can do this part without any calculus.)

Thus, combining the boundary information with the critical point information, we find that $f(x,y)$ has a minimum value of -1 (achieved on the boundary, at the points $(\cos 3\pi/4, \sin 3\pi/4)=(-1/\sqrt{2}, 1/\sqrt{2})$ and $(\cos 7\pi/4, \sin 7\pi/4)=(1/\sqrt{2}, -1/\sqrt{2})$) and a maximum value of $5/4$ (achieved at the interior points $(-\sqrt{3/8}, -\sqrt{3/8})$ and $(\sqrt{3/8}, \sqrt{3/8})$).

6.4.7ab (a) If we hold x constant at zero, then $f(0,y)=0$ for all y; therefore $f_2(0,y)=0$ for all y, and in particular, $f_2(0,0)=0$. Similarly, $f_1(0,0)=0$.

(b) If $x=y$ then $f(x,y)=\sqrt{|xy|}=\sqrt{|x^2|}=\sqrt{x^2}=|x|$. Thus if you look at the piece of the surface along the line $y=x$, it is a curve whose graph looks like the absolute value function: it has a corner at the origin. Such a curve does not have a tangent line at the origin, so the function $f(x,y)$ will not have a tangent plane at the origin.

6.5.8 First I'll compute the various partial derivatives of u:

$$\frac{\partial u}{\partial x}=F_1\frac{-y^2}{(xy)^2}+F_2\frac{-z^2}{(xz)^2},$$
$$\frac{\partial u}{\partial y}=F_1\frac{x^2}{(xy)^2}$$
$$\frac{\partial u}{\partial z}=F_2\frac{x^2}{(xz)^2}$$

We don't know what the function F is, so we have to leave F_1 and F_2 as unknown functions. Anyway, now we plug in to the formula in the problem:

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=F_1\frac{-y^2+y^2}{(xy)^2}+F_2\frac{-z^2+z^2}{(xz)^2}=0,$$

as desired.

6.5.15 We are told that "u is a function of r," so we have no way of simplifying $\frac{du}{dr}$. So let's work on the left side of the equation; let's start by computing the various partial derivatives:

$$\frac{\partial u}{\partial x}=\frac{du}{dr}\frac{\partial r}{\partial x}=\frac{du}{dr}x(x^2+y^2+z^2)^{-1/2}=\frac{du}{dr}\frac{x}{r}.$$

Similarly,

$$\frac{\partial u}{\partial y}=\frac{du}{dr}\frac{y}{r},\quad \frac{\partial u}{\partial z}=\frac{du}{dr}\frac{z}{r}.$$

Therefore

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{du}{dr}\frac{x}{r}\right)^2+\left(\frac{du}{dr}\frac{y}{r}\right)^2\left(\frac{du}{dr}\frac{z}{r}\right)^2$$
$$=\left(\frac{du}{dr}\right)^2\frac{x^2+y^2+z^2}{r^2}=\left(\frac{du}{dr}\right)^2,$$

as desired.