Mathematics 326 Midterm Exam

Instructions: This is a take-home exam. Do not consult with anyone about the exam except for Professor Palmieri. Explain all of your answers and write legibly. If you find a solution in a book or some other source, please provide a reference. Check your answers carefully; there will be no opportunity to turn in revisions for this exam.

Due in class on Friday, November 21, 2008.

1. Suppose that

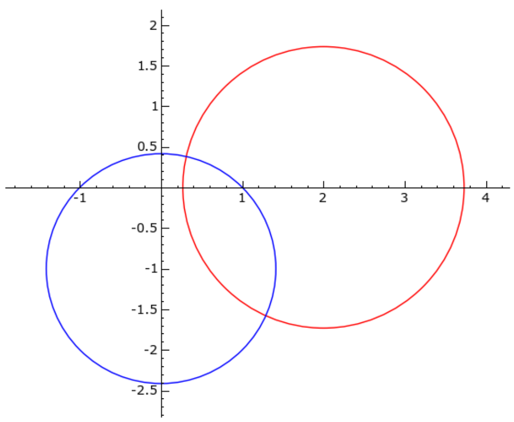
$$F(x, y, u, v) = x^2 + y^2 - 2ux + 1,$$
$$G(x, y, u, v) = x^2 + y^2 + 2vy - 1.$$

(a) (2 points) Interpret u and v as parameters and plot the curves $F = 0$, $G = 0$ in the xy-plane, assuming $u^2 \geq 1$.

Solution: Complete the square: $F = 0$ and $G = 0$ become, respectively,

$$(x - u)^2 + y^2 = u^2 - 1,$$
$$x^2 + (y + v)^2 = v^2 + 1.$$

The graphs of these are circles: the graph of $F = 0$ is a circle with radius $\sqrt{u^2 - 1}$ centered at $(u, 0)$, and the graph of $G = 0$ is a circle with radius $\sqrt{v^2 + 1}$ centered at $(0, -v)$. Here is a picture with $u = 2$ and $v = 1$:



(Note that if $u^2 = 1$, then the equation $F = 0$ has just one solution, $(u, 0)$, which you can think of as a circle of radius 0.)

(b) (3 points) Now suppose x_0, y_0, u_0, v_0 satisfy the equations $F = 0$, $G = 0$, and that $u_0^2 > 1$. Explain geometrically why it is reasonable to expect that, if u and v differ but slightly from u_0 and v_0, respectively, the equations $F = 0$, $G = 0$ will determine a *unique* point (x, y), if this point is required to be sufficiently near (x_0, y_0).

Solution: The end of the first sentence ("$u_0^2 > 1$") tells us that the $F = 0$ circle has positive radius. The rest of the first sentence tells us that we have a point which is a solution to the equations $F = 0$ and $G = 0$, so we have parameters u_0 and v_0 such that the circles intersect at a point (x_0, y_0). The situation could be as in the picture, where (x_0, y_0) is one of the two points of intersection.

Now, if (u, v) is close to (u_0, v_0), then the corresponding circles are close to the original circles. If the circles for (u_0, v_0) intersect at a point (x_0, y_0), and if we change the radii and centers just a little bit, then the new circles will intersect at a point near (x_0, y_0). The new circles might intersect at a second point, also, but that point won't be very close to (x_0, y_0); thus there will be a unique point (x, y) of intersection near (x_0, y_0).

We have to rule out one possible problem: if the circles for the parameters (u_0, v_0) were tangent, just intersecting at a single point, then changing (u_0, v_0) a little might result in no intersection points (bad: no solutions) or two intersection points (bad: no unique solution near the point $((x_0, y_0))$. The $G = 0$ circle always goes through the points $(1, 0)$ and $(-1, 0)$: just plug $x = \pm 1$, $y = 0$ into the equation for G. Also, as long as $u^2 > 1$, one of these points is inside the $F = 0$ circle: if $u > 1$, then $(1, 0)$ is inside the circle, and if $u < -1$, then $(-1, 0)$ is. Also note that the point $(0, -2v)$ is on the $G = 0$ circle but outside of the $F = 0$ circle. Therefore some points of the $G = 0$ circle are outside of the $F = 0$ circle, and some points of the $G = 0$ circle are inside of it. Thus the two circles cannot be tangent.

(c) (5 points) Show that a set of values x_0, y_0, u_0, v_0 cannot satisfy the three equations $F = 0$, $G = 0$, $\dfrac{\partial(F, G)}{\partial(x, y)} = 0$ unless $u_0^2 = 1$. Use this result and an appropriate version of the implicit function theorem for simultaneous equations to give an analytical explanation of the situation described in part (b).

Solution: By the work done in part (a), for the equation $F = 0$ to have any solutions at all, we need to assume that $u_0^2 \geq 1$.

The Jacobian J is equal to

$$J = \frac{\partial(F, G)}{\partial(x, y)} = \det \begin{bmatrix} 2x - 2u & 2y \\ 2x & 2y + 2v \end{bmatrix} = 4(xv - uy - uv).$$

Suppose that $F = 0$, $G = 0$, and $J = 0$ at some point (x_0, y_0, u_0, v_0). If we subtract the equation $F = 0$ equation from the $G = 0$ equation, we get $2v_0 y_0 + 2u_0 x_0 - 2 = 0$; this gives us these three equations:

$$\begin{aligned} x_0^2 + y_0^2 + 2v_0 y_0 - 1 &= 0, \\ x_0 v_0 - u_0 y_0 - u_0 v_0 &= 0, \\ v_0 y_0 + u_0 x_0 - 1 &= 0. \end{aligned}$$

Solving the last two for x_0 and y_0 gives

$$x_0 = \frac{u_0(1 + v_0^2)}{u_0^2 + v_0^2}, \quad y_0 = \frac{v_0(1 - u_0^2)}{u_0^2 + v_0^2}.$$

Plug these into the first equation and do a lot of algebra: you end up with

$$\frac{1}{(u_0^2+v_0^2)^2}\left(v_0^4+v_0^2(1+u_0^2)+u_0^2\right)(1-u_0^2)=0.$$

The first two factors are positive, so the only way for this to be zero is if $u_0^2=1$.

Therefore if $P=(x_0,y_0,u_0,v_0)$ is a point which satisfies $F=0$ and $G=0$ and also has $u_0^2>1$, then the Jacobian $\dfrac{\partial(F,G)}{\partial(x,y)}$ must be nonzero. Therefore in a neighborhood of the point P, it is possible in the equations $F=0$ and $G=0$ to solve for x and y in terms of u and v; stated differently, given u and v near u_0 and v_0, there is a unique point (x,y) near (x_0,y_0) which is a simultaneous solution to $F=0$ and $G=0$.

(This is problem 8 from section 8.3.)

2. Let $F(x,y,z)=(x+1)(y-1)\sqrt[3]{z}-3xyz^2+\sin 2y$ and consider the equation $F(x,y,z)=0$ in a neighborhood of the point $(0,0,0)$.

   (a) (8 points) Does the implicit function theorem say that it is possible to solve for x in terms of y and z? For y in terms of x and z? For z in terms of x and y?

Solution: To solve for x in terms of y and z, the implicit function theorem requires that $\partial F/\partial x$ be nonzero at $(0,0,0)$, and similarly for the other cases. So I need to compute the partial derivatives of F:

$$\begin{aligned}\frac{\partial F}{\partial x}&=(y-1)\sqrt[3]{z}-3yz^2,\\ \frac{\partial F}{\partial y}&=(x+1)\sqrt[3]{z}-3xz^2+2\cos 2y,\\ \frac{\partial F}{\partial z}&=\frac{1}{3}(x+1)(y-1)z^{-2/3}-6xyz.\end{aligned}$$

At the point $(0,0,0)$, therefore, we have

$$\frac{\partial F}{\partial x}=0,\quad \frac{\partial F}{\partial y}=2,\quad \frac{\partial F}{\partial z}\text{ is undefined.}$$

So according to the implicit function theorem, we can solve for y in terms of x and z, near the origin. We cannot tell, just from the theorem, about solving for x or for z. (The implicit function theorem does *not* say that if its hypotheses fail, then you can't solve. If the hypotheses fail, then the conclusions may or may not be true. So it is correct to say that we cannot tell whether it is possible to solve for z, for example, in terms of x and y; it is incorrect to conclude, just from the theorem, that we cannot solve for z.)

(Technically, actually, the implicit function theorem requires all of the partials to be continuous in a neighborhood of the point in question. Since $\partial F/\partial z$ is undefined, this hypothesis is not satisfied, so the theorem doesn't apply at all.)

(b) (2 points) Forget about the implicit function theorem. Can you solve for x in terms of y and z in a neighborhood of $(0,0,0)$?

Solution: Basic algebra turns the equation $F(x,y,z)=0$ into

$$x = \frac{-\sin 2y - (y-1)\sqrt[3]{z}}{(y-1)\sqrt[3]{z} - 3yz^2}.$$

When $y=z=0$, there is a zero in the denominator, so this fraction is undefined. Thus you cannot solve for x in terms of y and z near the origin. (Note also that the limit of the right side as $(y,z)\to(0,0)$ is undefined: if you approach $(0,0)$ along the y-axis – that is with points of the form $(y,0)$, then the fraction is undefined for all such points, and so the limit is undefined.)

3. (10 points) Fix a real number a and find the shortest distance from the point $(0,0,a)$ to the surface defined by $z=x^2-y^2$. Use the methods from Section 7.6 in your solution. (Problems 4 and 5 in Section 7.6 are similar.)

Solution: Note that if $a=0$, then $(0,0,a)$ is on the surface, and the distance is zero.

The square of the distance from $(0,0,a)$ to (x,y,x^2-y^2) is

$$F(x,y,z) = x^2 + y^2 + (x^2-y^2-a)^2.$$

We want to minimize this function, so we will find the critical points and use the second derivative test to find the minimum. To find the critical points, we set $\partial F/\partial x = 0$, $\partial F/\partial y = 0$:

$$\begin{aligned} 2x + 4x(x^2-y^2-a) &= 0, \\ 2y - 4y(x^2-y^2-a) &= 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} x(1+2(x^2-y^2-a)) &= 0, \\ y(1-2(x^2-y^2-a)) &= 0. \end{aligned}$$

Thus either $x=0$ or $1+2(x^2-y^2-a)=0$, and either $y=0$ or $1-2(x^2-y^2-a)=0$. This gives four possible combinations:

- If $x=0$ and $y=0$, we have a critical point, $(0,0)$.
- If $x=0$ and $1-2(x^2-y^2-a)=0$, then the second equation becomes $1/2 = -y^2-a$, or $y^2=-a-1/2$. Thus we have two critical points, $(0,\sqrt{-a-1/2})$ and $(0,-\sqrt{-a-1/2})$. These obviously only make sense if $-a-1/2\geq 0$, so if $a\leq -1/2$. (Note that if $a=-1/2$, then this just produces the point $(0,0)$ again.)
- Similarly, if $1+2(x^2-y^2-a)=0$ and $y=0$, then we get two critical points, $(\sqrt{a-1/2},0)$ and $(-\sqrt{a-1/2},0)$. These only make sense if $a\geq 1/2$. (As above, note that if $a=1/2$, then this just produces the point $(0,0)$ again.)

To identify the type of each, we use the second derivative test, which involves calculating the various second partials of F:

$$\begin{aligned} F_{11} &= 2 + 12x^2 - 4y^2 - 4a, \\ F_{12} &= -8xy, \\ F_{22} &= 2 - 4x^2 + 12y^2 + 4a. \end{aligned}$$

Since either $x = 0$ or $y = 0$ at each critical point, then $F_{12} = 0$ at each critical point.

At $(0, 0)$: $F_{11} = 2 - 4a$ and $F_{11}F_{22} - F_{12}^2 = (2 - 4a)(2 + 4a) = 4 - 16a^2$. Thus if $a^2 > 1/4$, then $4 - 16a^2$ is negative, and so this critical point is a saddle point. If $a^2 < 1/4$, then $F_{11} > 0$ and $F_{11}F_{22} - F_{12}^2 > 0$, so we have a minimum. The distance from this point to $(0, 0, a)$ is $|a|$.

At $(0, \pm\sqrt{-a - 1/2})$: $F_{11} = 2$ and $F_{11}F_{22} - F_{12}^2 = 16(-1-2a)$. If $a < -1/2$, then these are both positive, and we have a minimum. The distance from each of these points to $(0, 0, a)$ is $\sqrt{-a - 1/4}$.

Similarly, at $(\pm\sqrt{a - 1/2}, 0)$: $F_{11} = 4(2a - 1)$ and $F_{11}F_{22} - F_{12}^2 = 16(2a - 1)$. As long as $a > 1/2$, these are both positive, and we have a minimum. The distance from each of these points to $(0, 0, a)$ is $\sqrt{a - 1/4}$.

Summarizing: if $a < -1/2$, then there are three critical points: a saddle at $(0, 0)$ and local minima at $(0, \pm\sqrt{-a - 1/2})$. Thus the minimum distance is $\sqrt{-a - 1/4}$. If $-1/2 < a < 1/2$, then there is one critical point: a local minimum at $(0, 0)$. Thus the minimum distance is $|a|$. If $a > 1/2$, then there are three critical points: a saddle at $(0, 0)$ and local minima at $(\pm\sqrt{a - 1/2}, 0)$. Thus the minimum distance is $\sqrt{a - 1/4}$.

If $a = -1/2$ or if $a = 1/2$, then we seem to have problems, because $F_{11}F_{22} - F_{12}^2 = 0$ at the critical point $(0, 0)$. However, There must be a minimum because of geometric considerations, and it must occur at the critical point. The critical point has distance $|a|$ to $(0, 0, a)$, so this is the minimum distance.

4. (10 points) Fix real numbers a, b, and c. Write the Taylor series for the function $F(x, y) = axy + bx^2 + c$ at the point $(x, y) = (1, 2)$; that is, in terms of powers of $x - 1$ and $y - 2$. Is $F(x, y)$ equal to its Taylor series?

Solution: Compute the various partial derivatives of F:

$$\begin{gathered} F_1 = ay + 2bx, \quad F_2 = ax, \\ F_{11} = 2b, \quad F_{12} = a, \quad F_{22} = 0, \\ F_{111} = 0, \quad F_{112} = 0, \quad F_{122} = 0, \quad F_{222} = 0, \end{gathered}$$

and more generally, all of the partials of degree 3 and higher are zero. So we plug into the formula for the Taylor series to get this:

$$(2a + b + c) + (2a + 2b)(x - 1) + a(y - 2) + \frac{1}{2}\left(2b(x - 1)^2 + 2a(x - 1)(y - 2)\right).$$

To answer the second question: when you multiply this all out, you get back the original function $F(x, y)$. Thus $F(x, y)$ equals its Taylor series.