Mathematics 326 Midterm Exam Name: ______________________

October 20, 2008

Instructions: This is a closed book exam, no notes or calculators allowed. Please turn off all cell phones, pagers, etc.

1. (10 points) Fix positive numbers a, b, and c. Find the minimum of $x + y + z$ subject to the conditions that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ and $x, y, z > 0$.

Solution: Use the method of Lagrange multipliers: let $u = x + y + z + \lambda \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right)$, and set the various partial derivatives of u equal to zero:

$$0 = 1 - \lambda \frac{a}{x^2}, \quad 0 = 1 - \lambda \frac{b}{y^2}, \quad 0 = 1 - \lambda \frac{c}{z^2}.$$

Use the first equation to solve for λ: $\lambda = x^2/a$. Plug this into the second equation:

$$1 = \frac{bx^2}{ay^2} \quad \text{or} \quad y^2 = bx^2/a \quad \text{or} \quad y = \sqrt{\frac{b}{a}}x.$$

Similarly, the third equation gives

$$z = \sqrt{\frac{c}{a}}x.$$

Plug these values for y and z into the constraint equation $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$:

$$1 = \frac{a}{x} + b\sqrt{\frac{a}{b}}\frac{1}{x} + c\sqrt{\frac{a}{c}}\frac{1}{x}$$

Thus $x = a + \sqrt{ab} + \sqrt{ac}$. Our equations for y and z in terms of x then give $y = \sqrt{ab} + b + \sqrt{bc}$ and $z = \sqrt{ac} + \sqrt{bc} + c$. Thus the function $x + y + z$, subject to the above constraint, has one extreme point, and its value at this point is

$$a + b + c + 2\sqrt{ab} + 2\sqrt{ac} + 2\sqrt{bc}.$$

Why is this a minimum? Along the constraint surface $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$, we can make at least one of x, y, or z very large, which would make the function $x + y + z$ very large. Thus the function has no maximum.

(This doesn't show that the critical point is actually a minimum, but it's good enough reasoning for the timed part of the exam.)

2. Suppose that $F(s,t)$ is some function, and let $u = x^3 F\left(\frac{y}{x}, \frac{z}{x}\right)$.

   (a) (5 points) What is $\frac{\partial u}{\partial x}$?

Solution: This is a simple application of the product rule combined with the chain rule:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 3x^2 F\left(\frac{y}{x}, \frac{z}{x}\right) + x^3 \frac{\partial}{\partial x} F\left(\frac{y}{x}, \frac{z}{x}\right) \\
&= 3x^2 F\left(\frac{y}{x}, \frac{z}{x}\right) + x^3 F_1\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \left(-\frac{y}{x^2}\right) + x^2 F_2\left(\frac{y}{x}, \frac{z}{x}\right) \cdot \left(-\frac{z}{x^2}\right) \\
&= 3x^2 F\left(\frac{y}{x}, \frac{z}{x}\right) - xyF_1\left(\frac{y}{x}, \frac{z}{x}\right) - xzF_2\left(\frac{y}{x}, \frac{z}{x}\right)
\end{aligned}$$

   (b) (5 points) Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$.

Solution: First compute $\partial u/\partial y$ and $\partial u/\partial z$:

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^3 F_1(x,y,z)\frac{1}{x} = x^2 F_1, \\
\frac{\partial u}{\partial z} &= x^3 F_2(x,y,z)\frac{1}{x} = x^2 F_2.
\end{aligned}$$

(Here, F stands for $F\left(\frac{y}{x}, \frac{z}{x}\right)$, and similarly for F_1 and F_2.)

Therefore

$$\begin{aligned}
x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= x(3x^2F - xyF_1 - xzF_2) + y(x^2F_1) + z(x^2F_2) \\
&= 3x^3F - x^2yF_1 - x^2zF_2 + x^2yF_1 + x^2zF_2 \\
&= 3x^3F \\
&= 3x^3F\left(\frac{y}{x}, \frac{z}{x}\right) \\
&= 3u,
\end{aligned}$$

as desired.

3. (10 points) Let F, g, and h be functions. Suppose that $z = f(x,y)$ satisfies an equation of the form

$$F(g(x,y,z), h(x,y,z)) = 0.$$

Show that

$$\frac{\partial z}{\partial y} = -\frac{F_1 g_2 + F_2 h_2}{F_1 g_3 + F_2 h_3}.$$

Solution: We apply $\partial/\partial y$ to the equation $F(g(x,y,z), h(x,y,z)) = 0$, using the fact that z depends on y:

$$\begin{aligned} 0 &= F_1 \left(\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} \right) + F_2 \left(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial y} \right) \\ &= F_1 g_2 + F_2 h_2 + \frac{\partial z}{\partial y} \left(F_1 g_3 + F_2 h_3 \right). \end{aligned}$$

Now solve for $\partial z/\partial y$:

$$\frac{\partial z}{\partial y} = -\frac{F_1 g_2 + F_2 h_2}{F_1 g_3 + F_2 h_3}.$$

4. (10 points) Define a real-valued function $f(x,y)$ on the xy-plane by

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{xy}\right) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Does $f(x,y)$ have any points of discontinuity? Explain your answer.

Solution: It is discontinuous at all points $(x,0)$ where $x \neq 0$. It is continuous everywhere else.

At points where $xy \neq 0$, there are no problems: the function is defined and made up of familiar functions (sin, products, quotients) all of which are continuous wherever they are defined.

The points where $xy = 0$ fall into two families: the y-axis (where $x = 0$) and the x-axis (where $y = 0$). As (x,y) approaches a point on the y-axis, that is as (x,y) approaches $(0, y_0)$, the factor x^2 goes to zero, while the factor $\sin(1/xy)$ lies between -1 and 1. Thus the product of these factors goes to zero, and the function is continuous at points on the y-axis.

As (x,y) approaches a point $(x_0, 0)$ on the x-axis, the function $\sin(1/xy)$ has no limit – it oscillates between -1 and 1 – while x^2 approaches x_0^2. As long as x_0 is nonzero, the limit of the product doesn't exist: the function value oscillates between $-x_0^2$ and x_0^2, roughly speaking.