Mathematics 326 Final Exam Name: Answers
December 11, 2008

Instructions: This is a closed book exam, no calculators allowed. You may use one sheet of handwritten notes. Please turn off all cell phones, pagers, etc.

1. (7 points) Given differentiable real-valued functions $F(x,y)$, $g(s,t)$, and $h(s,t)$, let $G(s,t) = F(g(s,t), h(s,t))$. Suppose that $g(0,1) = 2$ and $h(0,1) = 3$. Use differentials to write down the approximate value of $G(0,1) - G(1/100, 99/100)$, expressing your answer in terms of partial derivatives of F, g, and h. Make sure to indicate where each partial derivative is to be evaluated.

Solution: The differential dG is approximately equal to $G(x_0 + dx, y_0 + dy) - G(x_0, y_0)$, and is given by the formula

$$-dG = -\left(\frac{\partial F}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial h}{\partial s}\right) ds - \left(\frac{\partial F}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial h}{\partial t}\right) dt.$$

In our case, we want to compute $G(0,1) - G(1/100, 99/100)$, so to reconcile it with the above form, we should use $-dG$ with $s = 0$, $t = 1$, $ds = 1/100$, and $dt = -1/100$. (We could also use dG with $ds = -1/100$ and $dt = 1/100$.)

Each of the partials of F should be evaluated at $(x,y) = (2,3)$, and the partials of g and h should be evaluated at $(s,t) = (0,1)$. Thus the answer could be written as

$$-\frac{1}{100}\left(\frac{\partial F}{\partial x}(2,3)\frac{\partial g}{\partial s}(0,1) + \frac{\partial F}{\partial y}(2,3)\frac{\partial h}{\partial s}(0,1)\right) + \frac{1}{100}\left(\frac{\partial F}{\partial x}(2,3)\frac{\partial g}{\partial t}(0,1) + \frac{\partial F}{\partial y}(2,3)\frac{\partial h}{\partial t}(0,1)\right).$$

2. (5 points) Is the set $A = \{(x,y) : x^2 = y^2\}$ a vector space? Justify your answer.

Solution: No. Geometrically, the set A consists of the lines $x = y$ and $x = -y$, and this is not a single line through the origin (or a plane through the origin, etc.). Algebraically, the points $(1,1)$ and $(1,-1)$ are in A, but their sum $(2,0)$ is not.

3. (7 points) Find the maximum and minimum of $4x - 3y$ subject to the constraint $x^2 + y^2 = 25$.

Solution: I'll use Lagrange multipliers, so let $u(x, y) = 4x - 3y + \lambda(x^2 + y^2)$. Then I need to solve the three equations

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad x^2 + y^2 = 25.$$

That is,

$$4 + 2\lambda x = 0, \quad -3 + 2\lambda y = 0, \quad x^2 + y^2 = 25.$$

The first equation gives $\lambda = -2/x$, and the second gives $y = 3/(2\lambda) = -3x/4$. So the third equation turns into

$$x^2 + \frac{9}{16}x^2 = 25, \quad \text{or} \quad x^2 = 16.$$

Thus $x = \pm 4$ and so from $y = -3x/4$ I get two critical points: $(4, -3)$ and $(-4, 3)$. Because we're working over closed and bounded set (the set of points on the constraint curve), the function has a max and min. Therefore one of these two points will be the max and one will be the min, so I plug them into the original function $f(x, y) = 4x - 3y$ to figure out which is which: $f(4, -3) = 25$ and $f(-4, 3) = -25$. Therefore the function has its maximum value of 25 at the point $(4, -3)$, and it has its minimum value of -25 at the point $(-4, 3)$.

4. (a) (4 points) Given a function $f : \mathbf{R}^2 \to \mathbf{R}$, suppose that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ both exist at $(x, y) = (1, 2)$. What further conditions guarantee that f is differentiable at $(1, 2)$?

Solution: The function $\partial f/\partial x$ should be defined in neighborhood of $(1, 2)$ and be continuous at $(1, 2)$. (Or the same, but for $\partial f/\partial y$.)
(There are many correct answers here, because I did not ask for the weakest possible condition. For example, requiring $\partial f/\partial x$ to be continuous in a neighborhood of $(1, 2)$ is correct, because it implies the previous condition.)

(b) (3 points extra credit) Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ which is not differentiable at $(1, 2)$ but whose partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(1, 2)$.

Solution: See the book (sections 6.4 and 7.1) for examples like this.

5. Let (x_0, y_0, u_0) satisfy the equations

$$f(x) + f(y) = F(u), \quad g(x) + g(y) = G(u),$$

and assume that all of the functions here have continuous derivatives.

(a) (5 points) State a sufficient condition for being able to solve for x and y in terms of u in a neighborhood of (x_0, y_0, u_0).

Solution: Rewrite the system of equations as

$$H(x, y, u) = 0, \quad K(x, y, u) = 0$$

by setting

$$H(x, y, u) = f(x) + f(y) - F(u), \quad K(x, y, u) = g(x) + g(y) - G(u).$$

Then by the implicit function theorem, the condition is that the Jacobian $\dfrac{\partial(H, K)}{\partial(x, y)}$ is nonzero:

$$\det \begin{bmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \\ \frac{\partial K}{\partial x} & \frac{\partial K}{\partial y} \end{bmatrix} \neq 0.$$

In this case, we can write the partials of H and K in terms of the derivatives of the functions f and g: for example, $\partial H/\partial x = f'(x)$. So at the point (x_0, y_0, u_0), the condition becomes

$$f'(x_0)g'(y_0) - f'(y_0)g'(x_0) \neq 0.$$

(b) (2 points) What is that sufficient condition in the case $f(x) = x^2$, $g(x) = x^3$?

Solution: (Note that if $f(x) = x^2$, then $f(y) = y^2$, and the same for g.)
We just compute the derivatives and plug into the above equation:

$$(2x_0)(3y_0^2) - (3x_0)^2(2y_0) \neq 0,$$

or

$$6x_0y_0(y_0 - x_0) \neq 0.$$

This translates into $x_0 \neq 0$, $y_0 \neq 0$, and $x_0 \neq y_0$.

6. The function $T(x,y) = \left(\frac{4}{3}x + \frac{2}{3}y, \frac{1}{3}x - \frac{4}{3}y\right)$ is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$. (Don't prove this.)

   (a) (3 points) Find the matrix for T with respect to the standard basis $\{(1,0),(0,1)\}$.

   Solution: We apply T to each of the basis vectors and write the answers in terms of those basis vectors:

$$T(1,0) = (4/3, 1/3) = \frac{4}{3}(1,0) + \frac{1}{3}(0,1),$$
$$T(0,1) = (2/3, -4/3) = \frac{2}{3}(1,0) + \left(-\frac{4}{3}\right)(0,1).$$

   The matrix then has the coefficients from the first equation in the first column and the coefficients from the second equation in the second column: the matrix is $\begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$.

   (b) (3 points) The set $\{(1,1),(2,-1)\}$ forms a basis for $\mathbf{R}^2$. (Don't prove this.) Find the matrix for T with respect to this basis.

   Solution: Same procedure as above:

$$T(1,1) = (2,-1) = 0(1,1) + 1(2,-1),$$
$$T(2,-1) = (2,2) = 2(1,1) + 0(2,-1).$$

   So the matrix is $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

   (c) (4 points) Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a continuously differentiable function, and write $(u,v) = f(x,y)$. Suppose that the derivative of f at the point $(-2,-5)$ is the linear transformation T defined above. Is it possible to solve for x and y in terms of u and v in a neighborhood of $(x_0,y_0) = (-2,-5)$?

   Solution: Yes: the linear transformation T is invertible (because, for example, each of the matrices above has nonzero determinant). Therefore by the inverse function theorem, f is invertible in a neighborhood of $(-2,-5)$.