Homework #1

Problem. We know that $H^*(\mathbf{R}P^\infty;\mathbf{F}_2) \cong \mathbf{F}_2[x]$ with x in dimension 1. Use the Cartan formula and the instability conditions to compute $\mathrm{Sq}^i(x^j)$ for all i and j.

This is due on Friday, October 12.

---

1 Acyclic carriers

This is taken from Munkres [Mun84, Section 13].

Definition 1.1. Fix a ring R. Suppose that $\mathscr{C}$ is a chain complex of R-modules, free with basis B. Let $\mathscr{C}'$ be a chain complex of R-modules. An *acyclic carrier* from $\mathscr{C}$ to $\mathscr{C}'$, relative to the basis B, is a function

$$\Phi : B \to \{\text{subcomplexes of } \mathscr{C}'\},$$

such that

- $\Phi(\sigma)$ is acyclic for each $\sigma \in B$, and
- for basis elements σ and τ, if the coefficient of τ is nonzero in $\partial(\sigma)$, then $\Phi(\tau)$ is a subcomplex of $\Phi(\sigma)$.

A chain map $f : \mathscr{C} \to \mathscr{C}'$ is *carried* by Φ if $f(\sigma) \in \Phi(\sigma)$ for each $\sigma \in B$.

Theorem 1.2. *Let $\mathscr{C}$ and $\mathscr{C}'$ be chain complexes over R with $\mathscr{C}$ free. Let Φ be an acyclic carrier from $\mathscr{C}$ to $\mathscr{C}'$. Then there is a chain map $f : \mathscr{C} \to \mathscr{C}'$ carried by Φ. Any two such are chain homotopic, and the chain homotopy is carried by Φ.*

2 Equivariant acyclic carriers

From Mosher and Tangora [MT68, Chapter 2].

Definition 2.1. Let π be a group and let $\mathbf{Z}[\pi]$ denote the group ring of π. Let $\mathscr{C}$ be a free chain complex of $\mathbf{Z}[\pi]$-modules with basis B, and let $\mathscr{C}'$ be a chain complex of $\mathbf{Z}[\pi]$-modules. A π-*equivariant acyclic carrier* from $\mathscr{C}$ to $\mathscr{C}'$ is a function

$$\Phi : B \to \{\text{subcomplexes of } \mathscr{C}'\},$$

such that

- $\Phi(\sigma)$ is acyclic for each $\sigma \in B$,
- for basis elements σ and τ, if the coefficient of τ is nonzero in $\partial(\sigma)$, then $\Phi(\tau)$ is a subcomplex of $\Phi(\sigma)$, and

- for $x \in \pi$ and $\sigma \in B$, we have the containment $x \cdot \Phi(\sigma) \subseteq \Phi(\sigma)$.

A π-equivariant chain map $f : \mathscr{C} \to \mathscr{C}'$ is *carried* by Φ if $f(\sigma) \in \Phi(\sigma)$ for every $\sigma \in B$.

Theorem 2.2. *Let π, $\mathscr{C}$, and $\mathscr{C}'$ be as above, and let Φ be a π-equivariant acyclic carrier from $\mathscr{C}$ to $\mathscr{C}'$. Then there is a π-equivariant chain map $f : \mathscr{C} \to \mathscr{C}'$ carried by Φ. Any two such are chain homotopic, and the chain homotopy is carried by Φ.*

3 The Eilenberg-Zilber theorem

We also need the Eilenberg-Zilber theorem. If X is a topological space, write $\mathcal{S}(X)$ for its singular chain complex.

Theorem 3.1. *For topological spaces X and Y, there are natural chain maps*

$$\mathcal{S}(X) \otimes \mathcal{S}(Y) \rightleftarrows \mathcal{S}(X \times Y)$$

which are chain-homotopy inverse to each other.

References

[MT68] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, Publishers, New York-London, 1968.

[Mun84] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.