Mathematics 424B/574B Autumn 2007

Study problems for the final

There will be some problems on the final which are based on some of these; there will also be some problems on the final which are not based on any of these.

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1. Exercises 3.6, 3.12, 3.18

2. Let $\{a_n\}$ be a sequence of real numbers, let $\{p_n\}$ be a sequence of positive integers, and let
$$c_n = \frac{p_1 a_1 + p_2 a_2 + \ldots + p_n a_n}{p_1 + p_2 + \ldots + p_n}.$$
Prove that if the sequence $\{a_n\}$ converges to some number a, then $\{b_n\}$ converges to a also.

3. Let $\sum_{n=0}^{\infty} a_n$ be a series. For each n, let $s_n = a_0 + \ldots + a_n$. Let $\{b_n\}$ be a sequence. Suppose that

   (i) the sequence $\{s_n b_{n+1}\}$ converges, and

   (ii) the series $\sum_{n=0}^{\infty} s_n(b_n - b_{n+1})$ converges.

   Prove that the series $\sum_{n=0}^{\infty} a_n b_n$ converges. (Theorem 3.41 might be helpful.) Use this to give another proof of Theorem 3.42, and also use it to solve exercise 3.8.

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1. Exercise 4.4

2. Given a function $f : X \to Y$, prove that f is continuous if and only if $f(\overline{E}) \subseteq \overline{f(E)}$ for every subset E of X.

3. Suppose that $f : X \to Y$ is continuous. Prove that if X is compact, then $f(X)$ is compact. Prove that if X is connected, then $f(X)$ is connected. (Only assume the definition of compactness and the characterization of continuity in terms of preimages of open sets being open. Can you do it with just the ε-δ definition of continuity?)

4. Suppose that $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous. Define $h : X \to \mathbf{R}$ by $h(x) = \min(f(x), g(x))$. Must h be continuous?

5. Is there a continuous function $f : (0,1) \cup (1,2) \to \mathbf{R}$ with $f(1-) = 0$ and $f(1+) = 3$? Is there a uniformly continuous function like this?