Mathematics 424/574 Exam Name: ______________________

October 26, 2007

Instructions: This is a closed book exam, no notes or calculators allowed. Justify all of your answers, unless the problem says otherwise. Unless otherwise specified, you may refer to and use any result from the book, homework, or in-class problems. This is a timed exam, so you may use abbreviations and symbols (such as "$\forall$"): as long as I can make sense of what you write without struggling too much, it's okay.

Notation: $\mathbf{R}$ is the set of real numbers, $\mathbf{Q}$ is the set of rational numbers, and $\mathbf{Z}$ is the set of integers.

1. Treat each set here as a subset of $\mathbf{R}^2$. Let $A = N_1(-1,0)$ (the neighborhood of radius 1 around the point $(-1,0) \in \mathbf{R}^2$), and let $B = N_1(1,0)$. Let $\overline{B}$ be the closure of B. Let $C = (A \cup B)'$ be the set of limit points of $A \cup B$. Let D be the set of all points in A whose first coordinate is rational:

$$D = \{(x,y) \in A : x \in \mathbf{Q}\}.$$

(a) (8 points) Fill in each box in the following table with "yes" or "no". You do not have to explain your answers here.

Solution:

Is the set	at most countable?	open?	closed?	compact?	connected?
A	no	yes	no	no	yes
$\overline{B}$	no	no	yes	yes	yes
$A \cup B$	no	yes	no	no	no
C	no	no	yes	yes	yes
D	no	no	no	no	no

(b) (4 points) Is the point $(0,0)$ in C? Is the point $(1,0)$ in C? Give *brief* explanations.

Solution: Both points are in C. The point $(0,0)$ is distance 1 from $(1,0)$, the center of the open disk B. Any neighborhood $N_r(0,0)$ of $(0,0)$ will contain points of the form $(\varepsilon,0)$ where $0 < \varepsilon < \min(r,2)$, and those points are in B; thus any neighborhood of $(0,0)$ contains points of B. Since B is a subset of $A \cup B$, this means that $(0,0)$ is a limit point of $A \cup B$.

The point $(1,0)$ is the center of the open disk B, and every neighborhood $N_r(1,0)$ of $(1,0)$ contains points $(1+\varepsilon,0)$ where $0 < \varepsilon < \min(r,1)$. These points are in B, hence in $A \cup B$, and so $(1,0)$ is in C.

2. (10 points) Fix a nonzero real number b. Let $E = \{b^n : n \in \mathbf{Z}\}$. Does $\inf E$ exist? If so, what is it? Does $\sup E$ exist? If so, what is it? Justify your answers.

   You may use the following fact (without proving it): if $b > 1$, then for every real number x, there is an integer n so that $b^n > x$.

Solution: There are cases, depending on b. If $b = 1$, then $b^n = 1$ for every $n \in \mathbf{Z}$. Thus in this case the set E is equal to $\{1\}$. Therefore $\inf E = 1 = \sup E$. (Clearly 1 is a lower bound for E, and no number larger than 1 is also a lower bound, so $\inf E = 1$. A similar argument works for $\sup E$.)

Now assume that $b > 1$. By the "fact," we see that E has no supremum: since for any x, there is an n with $b^n > x$, we can conclude that no number x is an upper bound for E, so E has no upper bound, and hence no least upper bound.

I claim that if $b > 1$, then $\inf E = 0$. When $n < 0$, b^n is defined to be $1/b^{-n}$, and if $n < 0$, then $-n > 0$. Since b^{-n} is positive for all n, its reciprocal is also positive. Thus $b^n > 0$ for all n, so 0 is a lower bound for E. To prove that it is the greatest lower bound, we use the "fact" again. For any $y > 0$, there is some N so that $b^N > 1/y$, which means that $y > b^{-N}$. Therefore no positive y can be a lower bound, so 0 is indeed the greatest lower bound.

If $0 < b < 1$, then since $b^n = (1/b)^{-n}$, we see that

$$\inf\{b^n : n \in \mathbf{Z}\} = \inf\{(1/b)^{-n} : n \in \mathbf{Z}\} = \inf\{(1/b)^n : n \in \mathbf{Z}\}.$$

The same formula works for the supremum, also. If $0 < b < 1$, then $1/b > 1$, and we have already computed the infinum and supremum of the right-hand set: we get $\inf E = 0$ and $\sup E$ does not exist.

If $b = -1$, then $E = \{1, -1\}$, and so $\inf E = -1$ and $\sup E = 1$.

If $b < 0$ and $b \neq -1$, then neither $\inf E$ nor $\sup E$ exists. In this situation, either b or $1/b$ is less than -1, and the positive powers of such a number will become arbitrarily large in magnitude, and either positive (if the power is even) or negative (if the power is odd). So by the fact, there will be no upper or lower bound.

3. (10 points) For any subset E of $\mathbf{R}^k$, let E' be the set of limit points of E. Prove that E' is closed, using the definitions and/or theorems from the book. (Don't use any of the exercises.)

Solution: To show that E' is closed, we must show that E' contains all of its limit points, so let q be a limit point of E'. Our goal is to show that q is in E'.

If q is already in E', there is nothing to prove, so we may assume that q is not in E'. Since q is a limit point of E', every neighborhood $N_r(q)$ of q contains a point of E', say $p \neq q$. Since this point p is in E', p is a limit point of E, so every neighborhood of p contains a point of E. In particular, if we let $\varepsilon = \min(d(p,q), r - d(p,q))$, then the neighborhood $N_\varepsilon(p)$ contains a point x of E. Note that since $d(x,p) < \varepsilon \leq d(p,q)$, x cannot equal q. Also,

$$
\begin{aligned}
d(x,q) &\leq d(x,p) + d(p,q) && \text{by the triangle inequality} \\
&< (r - d(p,q)) + d(p,q) \\
&= r,
\end{aligned}
$$

so $x \in N_r(q)$. This means that every neighborhood of q contains a point x of E other than q itself, so q is a limit point of E; that is, $q \in E'$. This is what we wanted to show.

4. (8 points) Is there a nonempty perfect subset of $\mathbf{R}$ containing no irrational numbers?

Solution: No: if E is a nonempty perfect subset of $\mathbf{R}$, then E is uncountable. If E contained no irrational numbers, then E would consist entirely of rationals: that is, E would be contained in $\mathbf{Q}$. But E is uncountable and $\mathbf{Q}$ is countable, so this cannot happen.

5. (10 points) Suppose that X is a compact metric space, and let E be a closed subset of X. Prove that E is compact. (Don't just quote a theorem from the book; prove this using the definition of compactness.)

Solution: Let $\{G_\alpha\}$ be an open cover of E. Since E is closed, its complement E^c is open (by a theorem in the book). Since $E \subseteq \bigcup_\alpha G_\alpha$, we see that

$$X = E \cup E^c \subseteq \left(\bigcup_\alpha G_\alpha \right) \cup E^c.$$

Thus $\{G_\alpha\} \cup \{E^c\}$ is an open cover for X. Since X is compact, this has a finite subcover:

$$X = G_1 \cup \cdots \cup G_n \cup E^c$$

for some sets $G_1, \ldots, G_n$ in the open cover. (The compactness of X guarantees the existence of a finite subcover; E^c may or may not be part of that subcover, but it doesn't hurt to include it.) Then we have

$$E \subset X = G_1 \cup \cdots \cup G_n \cup E^c$$

so

$$E \subset G_1 \cup \cdots \cup G_n.$$

Thus E has a finite subcover, and so E is compact.