Mathematics 424/574 Final Exam Name: Answers

December 12, 2007

Instructions: This is a closed book exam, no notes or calculators allowed. Justify all of your answers, unless the problem says otherwise. Unless otherwise specified, you may refer to and use any result from the book, homework, or in-class problems. This is a timed exam, so you may use abbreviations and symbols (such as "∀"): as long as I can make sense of what you write without struggling too much, it's okay.

Notation: **R** is the set of real numbers.

1. (10 points) Let $\{a_n\}$ be a sequence of real numbers, let $\{p_k\}$ be a sequence of positive real numbers so that the partial sums $\sum_{k=1}^{n} p_k \to \infty$, and let

$$c_n = \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}.$$

Prove that if the sequence $\{a_n\}$ converges to some number a, then $\{c_n\}$ converges to a also.

Solution: Fix $\varepsilon > 0$, and consider $c_n - a$:

$$\begin{aligned} c_n - a &= \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n} - a \\ &= \frac{p_1(a_1 - a) + \cdots + p_n(a_n - a)}{p_1 + \cdots + p_n}. \end{aligned}$$

There is an integer N so that if $n > N$, then $|a_n - a| < \varepsilon/2$. There is also an integer M so that if $n > M$, then

$$\left| \frac{p_1(a_1 - a) + \cdots + p_N(a_n - a)}{p_1 + \cdots + p_n} \right| < \varepsilon/2 :$$

the numerator here is fixed, and the denominator gets larger as $n \to \infty$, so the fraction can be made arbitrarily close to zero. Therefore for any $n > \max(N, M)$, we have

$$\begin{aligned} c_n - a &= \frac{p_1(a_1 - a) + \cdots + p_n(a_n - a)}{p_1 + \cdots + p_n} \\ &= \frac{p_1(a_1 - a) + \cdots + p_N(a_N - a) + p_{N+1}(a_{N+1} - a) + \ldots p_n(a_n - a)}{p_1 + \cdots + p_n} \\ &= \frac{p_1(a_1 - a) + \cdots + p_N(a_N - a)}{p_1 + \cdots + p_n} \\ &= \frac{p_1(a_1 - a) + \cdots + p_N(a_N - a) + p_{N+1}(a_{N+1} - a) + \ldots p_n(a_n - a)}{p_1 + \cdots + p_n} \\ &= \frac{p_1(a_1 - a) + \cdots + p_N(a_N - a)}{p_1 + \cdots + p_n} + \frac{p_{N+1}(a_{N+1} - a) + \cdots + p_n(a_n - a)}{p_1 + \cdots + p_n}. \end{aligned}$$

Now take absolute values: the first fraction is less than $\varepsilon/2$ since $n > M$. For the second fraction, we have

$$\begin{aligned}\left|\frac{p_{N+1}(a_{N+1}-a)+\cdots+p_n(a_n-a)}{p_1+\cdots+p_n}\right| &\leq \frac{p_{N+1}|a_{N+1}-a|+\cdots+p_n|a_n-a|}{p_1+\cdots+p_n} \\ &< \frac{p_{N+1}\varepsilon/2+\cdots+p_n\varepsilon/2}{p_1+\cdots+p_n} \\ &< \frac{p_1\varepsilon/2+\cdots+p_n\varepsilon/2}{p_1+\cdots+p_n} = \varepsilon/2.\end{aligned}$$

Therefore for $n > \max(M,N)$, we have $|c_n - a| < \varepsilon$. Since ε was arbitrary, we conclude that $c_n \to a$.

2. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
   (a) (5 points) $a_n = \frac{n}{n^3+1}$

Solution: We rewrite a_n as $\frac{1}{n^2+\frac{1}{n}}$. Since $n^2 + \frac{1}{n} > n^2$, we have $\frac{1}{n^2+\frac{1}{n}} < \frac{1}{n^2}$. Everything here is positive, so we have $|a_n| < \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges, so by the comparison test, so does $\sum a_n$.

   (b) (5 points) $a_n = \begin{cases} \frac{(-1)^{n/2}}{2^{3n+4}} & \text{if } n \text{ is even,} \\ \frac{1}{3^{2n}} & \text{if } n \text{ is odd.} \end{cases}$

Solution: There are various ways to do this.
For example, we can use the root test. We have

$$\sqrt[n]{|a_n|} \to \begin{cases} \frac{1}{8} & \text{if } n \text{ is even,} \\ \frac{1}{9} & \text{if } n \text{ is odd.} \end{cases}$$

(For the case when n is even, the sign is of course irrelevant. The nth root of the denominator $2^{3n+4} = 2^{3n}2^4$ is

$$\sqrt[n]{2^{3n}}\sqrt[n]{2^4} = 2^3\sqrt[n]{2^4}.$$

As n goes to infinity, $\sqrt[n]{2^4} = 2^{4/n}$ goes to 1, so the limit is 8.)
Therefore $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \dfrac{1}{8}$, so the root test tells us that $\sum a_n$ converges.
Alternatively, we can break this into two sums, the even terms and the odd terms. If we can show that each of these converges, then the whole thing will. Each of these is a geometric series with ratio less than one, so they converge.

3. Let $f : X \to Y$ be a function between metric spaces.

   (a) (5 points) Prove that f is continuous if and only if $f^{-1}(U)$ is open for every open subset U of Y. (Use only the definition of continuity to do this – don't just cite theorems from the book.)

   Solution: First suppose that f is continuous, and suppose that U is open in Y. Fix $x \in f^{-1}(U)$; we want to show that x is an interior point of $f^{-1}(U)$. Since $x \in f^{-1}(U)$, we know that $f(x) \in U$. Since U is open, there is an $\varepsilon > 0$ so that $N_\varepsilon(f(x)) \subseteq U$. Furthermore, since f is continuous at x, given this ε, there exists a $\delta > 0$ so that if $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. That is, if $d_X(x,y) < \delta$, then $f(y) \in N_\varepsilon(f(x)) \subseteq U$, which means that $y \in f^{-1}(U)$. Therefore the δ-neighborhood of x is contained in $f^{-1}(U)$, so x is an interior point, and $f^{-1}(U)$ is open.

   Now suppose that the preimage of every open set is open; choose a point $x \in X$, and fix $\varepsilon > 0$. Then $N_\varepsilon(f(x))$ is open in Y, so its preimage is open in X. Since x is in this preimage, it must be an interior point, so there is a $\delta > 0$ with $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x)))$. But this means that for every $y \in X$ with $d_X(x,y) < \delta$, we have $f(y) \in N_\varepsilon(f(x))$, so $d_Y(f(x), f(y)) < \varepsilon$. Therefore f is continuous at x. Since x was an arbitrary point of X, we conclude that f is continuous everywhere.

   (b) (5 points) Prove that f is continuous if and only if $f^{-1}(V)$ is closed for every closed subset V of Y. (Use only the definition of continuity and the result from part (a).)

   Solution: First suppose that f is continuous. Suppose that V is closed in Y. Then its complement V^c is open in Y, and $f^{-1}(V^c) = (f^{-1}(V))^c$. By part (a), $f^{-1}(V^c)$ is open, and therefore its complement $f^{-1}(V)$ is closed.

   Now suppose that the preimage of any closed set is closed. We will show that the preimage of any open set is open, so suppose that U is an open subset of Y. Then U^c is closed, and again $f^{-1}(U^c) = (f^{-1}(U))^c$; by assumption, $f^{-1}(U^c)$ is closed, so its complement $f^{-1}(U)$ is open. Therefore f is continuous.

(Problem 3, continued)

   (c) (5 points) Suppose that f is continuous. Prove that if X is connected, then $f(X)$ is connected. (Use only the definition of connectedness, the definition of continuity, and the results from parts (a) and (b).)

   Solution: Suppose that there is a separation of $f(X)$: suppose that $f(X) = A \cup B$ with $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, with A and B nonempty. Then $f^{-1}(A)$ and $f^{-1}(B)$ are both nonempty, $f^{-1}(\overline{A}) \cap f^{-1}(B) = f^{-1}(\overline{A} \cap B)$ and hence is empty, and similarly for $f^{-1}(A) \cap f^{-1}(\overline{B})$. Also, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$. Finally, $f^{-1}(\overline{A})$ is closed and contains $f^{-1}(A)$, and hence contains $\overline{f^{-1}(A)}$. Taken together, this means that $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X.

   This proves the contrapositive: if $f(X)$ is not connected, then X is not connected.

4. (a) (5 points) Is there a continuous function $f : (0,1) \cup (1,2) \to \mathbf{R}$ with $f(1-) < f(1+)$?

> **Solution:** Yes: the function f defined by
>
> $$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ 2 & \text{if } 1 < x < 2 \end{cases}$$
>
> is continuous: given any x in the domain and any $\varepsilon > 0$, choose δ so that $\delta < |x-1|$. Then if $|y-x| < \delta$, then x and y are both in the same "piece" of the domain: if $x < 1$, then $y < 1$, while if $x > 1$, then $y > 1$. Therefore $f(x) = f(y)$, so $|f(y) - f(x)| = 0$ is less than ε.

(b) (5 points) Is there a uniformly continuous function $f : (0,1) \cup (1,2) \to \mathbf{R}$ with $f(1-) < f(1+)$?

> **Solution:** No: pick ε with $f(1+) - f(1-) > 3\varepsilon > 0$. Then there is no choice of δ which makes the function satisfy the definition of uniform continuity: for any $\delta > 0$, there is an x with $1 - \frac{\delta}{2} < x < 1$ and $|f(x) - f(1-)| < \varepsilon$ – this is because the limit $f(1-)$ exists – and similarly there is a y with $1 < y < 1 + \frac{\delta}{2}$ and $|f(y) - f(1+)| < \varepsilon$. Therefore we have $f(x) < f(1-) + \varepsilon$ and $f(y) > f(1+) - \varepsilon$, so the distance between the points x and y is less than δ, while
>
> $$|f(y) - f(x)| = f(y) - f(x) < (f(1+) - \varepsilon) - (f(1-) + \varepsilon) = (f(1+) - f(1-)) - 2\varepsilon < 3\varepsilon - 2\varepsilon = \varepsilon.$$
>
> Thus f cannot be uniformly continuous.

5. Let X be a metric space, and suppose that $f : X \to \mathbf{R}$ is continuous.

(a) (5 points) Assume that X is compact, and prove that there exist points p and q in X such that $f(p) \le f(x) \le f(q)$ for all $x \in X$.

> **Solution:** This is just the extreme value theorem.
>
> Since X is compact and f is continuous, the image $f(X)$ is compact in $\mathbf{R}$, and hence is closed and bounded. Since $f(X)$ is bounded, it has a sup and an inf. Since it is closed, it contains its sup and inf; therefore there are points $p, q \in X$ so that $f(p) = \inf f(X)$ and $f(q) = \sup f(X)$. This is precisely what was to be proved.

(b) (5 points) Show that if X is not compact, then the conclusion in part (a) need not hold.

> **Solution:** Let $X = \mathbf{R}$, and let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x$. Then f is continuous (the identity function is always continuous), and since $\mathbf{R}$ is not bounded, there are no points $p, q \in \mathbf{R}$ so that $p \le x \le q$ for all $x \in \mathbf{R}$.