2.12. Let K be the following subset of $\mathbf{R}$:

$$K = \{0\} \cup \left\{ \frac{1}{n} : n = 1, 2, 3, \dots \right\}.$$

Prove that K is compact directly from the definition.

Let $\{G_\alpha\}$ be an open cover of K. I have to show that there is a finite subcover. Since this is an open cover, $K \subseteq \bigcup_\alpha G_\alpha$, so in particular, the point 0 is in one of the sets in the cover, say G_{α_0}. Since G_{α_0} is open, every point in it is an interior point, and in particular, 0 is an interior point. So G_{α_0} contains a neighborhood of the form $N_\epsilon(0)$ for some $\epsilon > 0$. Since $\epsilon > 0$, there is an n so that $\epsilon > 1/n$ (by the archimedean property of $\mathbf{R}$, Theorem 1.20(a), with $x = \epsilon$ and $y = 1$). Therefore the neighborhood contains all points x on the real line with $|x| \leq 1/n$, so the points

$$\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots$$

are all contained in $N_\epsilon(0) \subseteq G_{\alpha_0}$.

Now I'm ready to exhibit a finite subcover of $\{G_\alpha\}$. For each number $i = 1$, 2, $\dots$, $n-1$, find an open set G_{α_i} from the cover which contains $1/i$. Then K is contained in

$$G_{\alpha_0} \cup G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_{n-1}}.$$

Why? For $i = 1, \dots, n-1$, the point $1/i$ is in G_{α_i}. The rest of the points in K are in G_{α_0}. Therefore

$$\{G_{\alpha_0}, G_{\alpha_1}, \dots, G_{\alpha_{n-1}}\}$$

is a finite subcover. I have showed that every open cover of K contains a finite subcover, and therefore K is compact.