Mathematics 412

1 February 2006
Midterm preview

Instructions: For this exam, clarity of exposition is as important as correctness of mathematics.

The actual exam will be closed book, no notes or calculators allowed. There will be room on the paper to write your answers. Since it is a timed exam, you can use abbreviations and shorthand, and you don't need to use complete sentences, as long as I can easily understand what you're saying.

1. A friend comes to you and asks if a particular polynomial $p(x)$ of degree 25 in $\mathbb{F}_2[x]$ is irreducible. The friend explains that she has tried dividing $p(x)$ by every polynomial in $\mathbb{F}_2[x]$ of degree from 1 to 18 and has found that $p(x)$ is not divisible by any of them. She is getting tired of doing all these divisions and wonders if there's an easier way to check whether or not $p(x)$ is irreducible. You surprise your friend with the statement that she need not do any more work: $p(x)$ is indeed irreducible!

   Prove this; that is, use the fact that no polynomial of degree between 1 and 18 divides $p(x)$ to prove that $p(x)$ is irreducible. Do not simply quote a theorem that makes this problem trivial; rather, provide an argument "from scratch" using the given information. You may use the fact that the degree of a product of two polynomials is the sum of the degrees of the two polynomials.

2. A famous theorem of Gauss's says:

   > Every irreducible polynomial in $\mathbb{R}[x]$ has degree either one or two.

   Use this to prove:

   > Every polynomial of odd degree in $\mathbb{R}[x]$ has at least one real root.

3. For your answers to this question, do not simply quote and apply a major theorem. Rather, give proofs from scratch.

   (a) Must the polynomial $x^n - 14$ be irreducible in $\mathbb{Q}[x]$ for every $n \geq 1$? Justify your answer.

   (b) Must the polynomial $x^n - 49$ be irreducible in $\mathbb{Q}[x]$ for every $n \geq 1$? Justify your answer.

4. Prove that the polynomial
   $$15x^4 + 7x^3 - 4x^2 - 33$$
   does not factor in $\mathbb{Z}[x]$ as the product $g(x)h(x)$ of two polynomials $g(x)$ and $h(x)$ whose degrees are both less than 4. (You may use theorems for this problem, as long as you explain what you're using.)

5. Let K be a field.

   (a) State Bezout's Theorem for a pair of polynomials $a(x)$ and $b(x)$ in $K[x]$.

   (b) Prove the statement below.

      Suppose that $a(x)$ and $b(x)$ are relatively prime polynomials in $K[x]$ and $a(x)$ divides the product $b(x)c(x)$ in $K[x]$. Then $a(x)$ divides $c(x)$.

   You may use Bezout's theorem in your proof. If you do, be sure to make clear where and how you are using it.

   (c) Use the Euclidean algorithm to find a greatest common divisor of $x^3 + 1$ and $x^5 + 1$ in $\mathbb{F}_3[x]$.