Name: Answers

Instructions: This is a closed book quiz, no notes or calculators allowed. Please check your answers carefully; I will only award limited partial credit.

1. (6 points) For each part, give a brief geometric description of the given subset of $\mathbf{R}^2$, and say whether it is a subspace or not.

   (a) $\{(x,y) : x^2+y^2=1\}$

   Solution: This represents a unit circle centered at the origin. This is **not a subspace**: from the geometric point of view, it is not a line or plane through the origin. It also fails *all* of the parts of the definition of a subspace that I gave in class: it does not contain the origin, it is not closed under vector addition, and it is not closed under scalar multiplication.

   (b) $\{(x,y) : -3x+y=2\}$

   Solution: This is a line with y-intercept 2 and slope 3. So it does not pass through the origin, and it is **not a subspace**. It also fails all of the parts of the definition of a subspace that I gave in class: it does not contain the origin, it is not closed under vector addition, and it is not closed under scalar multiplication.

2. (4 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$.

   Do either part (a) or part (b). DO NOT DO BOTH. Clearly indicate which one you're doing. For either part, don't just state an answer: explain what you're doing.

   (a) Give an algebraic specification for the null space of A.

   Solution: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, set $A\mathbf{x}$ equal to zero, and solve for $\mathbf{x}$:

   $$A\mathbf{x} = \begin{bmatrix} x_1+2x_2+3x_3 \\ x_2+2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Now solve this system of equations for $\mathbf{x}$. We can row reduce, or we can just plug and chug. For example, the second equation says that $x_2 = -2x_3$, and plugging this into the first equation gives $x_1 - 4x_3 + 3x_3 = 0$, so $x_1 = x_3$. So the null space is

   $$\{\mathbf{x} \text{ in } \mathbf{R}^3 : x_1 = x_3,\ x_2 = -2x_3\}.$$

   Another description: all scalar multiples of the vector $(1,-2,1)$:

   $$\{(1,-2,1)t \text{ for all } t\}.$$

(b) Give an algebraic specification for the range of A.

Solution: The question is, for which vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is there a solution to the equation $A\mathbf{x} = \mathbf{b}$? So row reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \end{bmatrix}.$$

This takes one step: its reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & b_1 - 2b_2 \\ 0 & 1 & 2 & b_2 \end{bmatrix}.$$

So the solution is: $x_1 - x_3 = b_1 - 2b_2$ and $x_2 + 2x_3 = b_2$. This can be solved for any b_1 and b_2, so the range is all of $\mathbf{R}^2$.

Alternatively, the range of A is equal to the column space of A, which equals the row space of A^T. To get a simpler description of this, row reduce A^T:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So the row space of A^T equals the span of $(1,0)$ and $(0,1)$, which is all of $\mathbf{R}^2$.