Name: Answers

Instructions: This is a closed book exam, no calculators allowed. You may use one two-sided sheet of handwritten notes. Please check your answers carefully; I will only award limited partial credit. If you need more room, use the backs of the pages, and indicate that you have done so.

1. (20 points) Let A be the following 4×4 matrix:

$$A = \begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 2 & 5 & -10 \\ 1 & 4 & 8 & -6 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

Find an orthonormal basis for the range of A.

Solution: The method is: first find a spanning set for the range of A, then find a basis, then use Gram-Schmidt to turn it into an orthogonal basis, and then normalize each vector to get an orthonormal basis.

The range of A is spanned by the columns of A, so they form a spanning set. To get a basis, we take the transpose of A, row reduce, transpose again, and keep the nonzero columns (equivalently, transpose, row reduce, and keep the nonzero rows):

$$A \xrightarrow{A^t} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & 2 & 4 & -3 \\ 4 & 5 & 8 & -1 \\ 2 & -10 & -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 5 & 4 & -1 \\ 0 & -10 & -8 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 4 & 13/2 \\ 0 & 0 & -8 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 13/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This isn't quite in reduced echelon form, because of the entry above the 1 in the third row, but changing that won't change which rows are nonzero, and as it stands now, the first two rows are already orthogonal. So a basis for the range is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 13/8 \end{bmatrix} \right\}$ We appply the Gram-Schmidt procedure to this. As I mentioned earlier, the first two are already orthogonal, so call them $\mathbf{u}_1$ and $\mathbf{u}_2$. Nowi just have to replace the third (which I'll call $\mathbf{w}_3$) with

$$\mathbf{w}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{w}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{w}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 13/8 \end{bmatrix} - 1/2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 3/4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 3/4 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

So an orthogonal basis for the range is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 3/4 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}$. Finally, to make this orthonormal, just divide each vector by its length: an orthonormal basis for the range is

$$\left\{ 1/\sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, 2/\sqrt{13} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3/2 \end{bmatrix}, 4/\sqrt{21} \begin{bmatrix} -1/2 \\ 3/4 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}.$$

(There are actually infinitely many correct answers.)

Another approach: you can actually do Gram-Schmidt with a spanning set, rather than a basis. So apply the Gram-Schmidt process to the columns of the matrix. One of the outputs is the zero vector, so discard that and keep the other three; this will be an orthogonal basis. Then divide by the lengths to get an orthonormal basis.

2. Short answer questions. Give a (brief) justification for each answer.

   (a) (5 points) Consider the subspace $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + 2x_2 - x_3 = 0,\ x_2 + 3x_3 = 0 \right\}$. What is the dimension of W? (You don't need to explain why this is a subspace.)

   Solution: Its dimension is 1. A brief reason for this is that the vectors in W depend on just one parameter, x_3. In more detail, each vector in W is of the form $\begin{bmatrix} 7x_3 \\ -3x_3 \\ x_3 \end{bmatrix}$, so W is spanned by $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$. This single vector forms a basis for W, so W is 1-dimensional.

   (b) (5 points) Define a function $F : \mathbf{R}^2 \to \mathbf{R}^3$ by $F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2 \\ x_2 \end{bmatrix}$. Is F a linear transformation?

   Solution: No, it is not. For example, $F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $F\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, so $F\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \neq 2F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

(c) (5 points) Suppose that A is a 4×3 matrix, and suppose that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A. What is the rank of A?

> **Solution:** Its rank is 1. The rank-nullity theorem says that rank+nullity=number of columns. This matrix has 3 columns, and its null space is 2-dimensional, so its nullity is 2.

3. (5 points) For which values of a, b, and c is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 1 \end{bmatrix} \right\}$ an orthogonal set?

> **Solution:** In order for the first two to be orthogonal, their dot product must be zero: $a + 6 = 0$. Therefore $\boxed{a = -6}$. In order for the first and third to be orthogonal, we must have $b + 3 = 0$, so $\boxed{b = -3}$. The dot product of the second and third is $= 0ab + 2c + 2 = 18 + 2c + 2 = 20 + 2c$. Therefore $\boxed{c = -10}$.

4. (10 points) Let $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$. Is it true that every vector in the range of A is also in the null space of A? Explain your answer.

> **Solution:** Method: get a description of the range, and then test to see if the vectors in the range are also in the null space.
>
> The range of A is the column space, so the range is Span $\left(\begin{bmatrix} 3 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right)$. These two vectors are scalar multiples of each other, so I can say that the range is Span $\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix} \right)$. Therefore the vectors in the range are all of the form $\begin{bmatrix} -a \\ -3a \end{bmatrix}$. Are such vectors in the null space? Multiply by A:
>
> $$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} -a \\ -3a \end{bmatrix} = \begin{bmatrix} -3a + 3a \\ -9a + 9a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
>
> So, yes, every vector in the range is in the null space.
>
> Alternatively, the clever and devious way to do the problem is to notice that $A^2 = 0$, and that this implies that the range is contained in the null space.