Name: Answers

Instructions: This is a closed book exam, no notes or calculators allowed. Please check your answers carefully; I will only award limited partial credit. If you need more room, use the backs of the pages, and indicate that you have done so.

1. Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

(a) (10 points) Is A invertible? If so, find its inverse.

Solution: Form the augmented matrix

$$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and row-reduce it. The result is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 7 & 1 \end{bmatrix}.$$

Since the left half of this is the identity matrix, A is invertible, and its inverse is the right half:

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ -2 & 7 & 1 \end{bmatrix}.$$

(Now check it: multiply A by this matrix and see if you get the 3×3 identity matrix.)

(b) (5 points) Is A singular? Is A nonsingular?

Solution: A is nonsingular (because it's invertible). It is not singular.

2. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} a \\ 3 \\ 5 \end{bmatrix}.$$

(a) (10 points) For which values of a is $\{v_1, v_2, v_3\}$ linearly independent?

Solution: I'll do this like Example 2 on page 73 of the book: Make a matrix with these vectors as the columns, add in a column of zeroes, and try to row-reduce the result:

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 2 & 0 & 3 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 0 & 3-2a & 0 \\ 0 & 1 & 5-3a & 0 \end{bmatrix}$$

If $3 = 2a$, which is to say if $a = 3/2$, then this has a row of all zeroes, and the solution will be: x_3 is arbitrary, $x_1 + ax_3 = 0$, and $x_2 + (5-3a)x_3 = 0$. Thus there are infinitely many solutions, so the vectors are linearly dependent.

If, on the other hand, $a \neq 3/2$, then the matrix is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which means that they are linearly independent. So the answer is: $a \neq 3/2$.

(b) (5 points) For which values of a is $\{v_1, v_2, v_3\}$ linearly dependent?

Solution: The vectors are linearly dependent when $a = 3/2$.

3. (10 points) Find the interpolating polynomial for the data

t	0	1	2
y	2	2	3

Solution: We're looking for a formula like $y = at^2 + bt + c$ (degree 2 since there are three points specified). Plug in the three points (0,2), (1,2), and (2,3) to get

$$\begin{array}{ccccccc} & & & & c & = & 2 \\ a & + & b & + & c & = & 2 \\ 4a & + & 2b & + & c & = & 3 \end{array}$$

You can solve this by row-reduction, or just by playing with the equations. Plugging $c = 2$ into the second equation gives $a + b = 0$, so $b = -a$. Plugging this into the last equation gives $4a - 2a + 2 = 3$, so $a = 1/2$. So the formula is $y = \frac{1}{2}t^2 - \frac{1}{2}t + 2$.

(Now check it: plug in $t = 0$, $t = 1$, and $t = 2$, and see if you get 2, 2, and 3, respectively.)

4. (10 points) Find all α and β which satisfy

$$\begin{aligned} 4\alpha^2 - 2\beta^2 &= 1 \\ -3\alpha^2 + 2\beta^2 &= 5 \end{aligned}$$

Solution: Let $x = \alpha^2$ and $y = \beta^2$. Then I have a system of linear equations

$$\begin{aligned} 4x - 2y &= 1 \\ -3x + 2y &= 5 \end{aligned}$$

You can solve this by whatever means you like; the result is $x = 6$ and $y = 23/2$. Therefore $\alpha = \pm\sqrt{6}$ and $\beta = \pm\sqrt{23/2}$.

(Check it, either after finding x and y by plugging them in, or by plugging α and β into the original equations.)

5. (a) (10 points) Put the matrix

$$\begin{bmatrix} 0 & 0 & 2 & 8 & 4 \\ 0 & 1 & 3 & 11 & 9 \\ 3 & -3 & -9 & -24 & -33 \\ -2 & 1 & 6 & 17 & 21 \end{bmatrix}$$

into reduced row echelon form.

Solution: Okay, here we go:

$$\begin{bmatrix} 0 & 0 & 2 & 8 & 4 \\ 0 & 1 & 3 & 11 & 9 \\ 3 & -3 & -9 & -24 & -33 \\ -2 & 1 & 6 & 17 & 21 \end{bmatrix} \xrightarrow{\mathbf{R}_1 \leftrightarrow \mathbf{R}_3} \begin{bmatrix} 3 & -3 & -9 & -24 & -33 \\ 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 2 & 8 & 4 \\ -2 & 1 & 6 & 17 & 21 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}\mathbf{R}_1} \begin{bmatrix} 1 & -1 & -3 & -8 & -11 \\ 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 2 & 8 & 4 \\ -2 & 1 & 6 & 17 & 21 \end{bmatrix}$$

$$\xrightarrow{\mathbf{R}_4 + 2\mathbf{R}_1} \begin{bmatrix} 1 & -1 & -3 & -8 & -11 \\ 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 2 & 8 & 4 \\ 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{\mathbf{R}_1 + \mathbf{R}_2,\ \mathbf{R}_4 + \mathbf{R}_2} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 2 & 8 & 4 \\ 0 & 0 & 3 & 12 & 8 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\mathbf{R}_3} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 3 & 11 & 9 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 3 & 12 & 8 \end{bmatrix}$$

$$\xrightarrow{\mathbf{R}_2 - 3\mathbf{R}_3,\ \mathbf{R}_4 - 3\mathbf{R}_3} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\mathbf{R}_4} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This last matrix is in reduced row-echelon form.

(Problem 5, continued)

(b) (5 points) Solve the system

$$\begin{array}{rcrcrcrcr} & & & & 2x_3 & + & 8x_4 & = & 4 \\ & & x_2 & + & 3x_3 & + & 11x_4 & = & 9 \\ 3x_1 & - & 3x_2 & - & 9x_3 & - & 24x_4 & = & -33 \\ -2x_1 & + & x_2 & + & 6x_3 & + & 17x_4 & = & 21 \end{array}$$

Solution: The matrix in part (a) is exactly the augmented matrix for this system. The last row of that matrix corresponds to the equation $0 = 1$, so the system has no solutions.

(c) (5 points) Is this system consistent or inconsistent?

Solution: The system is inconsistent.

6. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Find all b so that $AB = BA$.

Solution: Well,

$$AB = \begin{bmatrix} 1 & 6 & 10+2b \\ 0 & 1 & b+2 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 1 & 6 & 18 \\ 0 & 1 & b+2 \\ 0 & 0 & 1 \end{bmatrix}.$$

These only differ in the upper right corner, so for them to be equal, we must have $10 + 2b = 18$, which means that $b = 4$.