Solutions, homework assignment 7

Problem from web page: Suppose that $A\mathbf{x} = \lambda\mathbf{x}$. Let $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$. Then

$$\begin{aligned} f(A)\mathbf{x} &= \left(a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k\right)\mathbf{x} \\ &= a_0 I\mathbf{x} + a_1 A\mathbf{x} + a_2 A^2\mathbf{x} + \cdots + a_k A^k\mathbf{x} \end{aligned}$$

Now apply Theorem 11(a) to compute $A^i\mathbf{x}$ for each i:

$$\begin{aligned} &= a_0\mathbf{x} + a_1\lambda\mathbf{x} + a_2\lambda^2\mathbf{x} + \cdots + a_k\lambda^k\mathbf{x} \\ &= \left(a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_k\lambda^k\right)\mathbf{x} \\ &= f(\lambda)\mathbf{x}, \end{aligned}$$

as desired.

Section 4.4: 24: (a) If B is a 3×3 matrix satisfying $B\mathbf{x} = \mathbf{0}$ for every vector $\mathbf{x}$, then B must be the zero matrix: the columns of B are equal to $B\mathbf{e}_i$, for $i = 1, 2, 3$, so each column is equal to $\mathbf{0}$, so the whole matrix is filled with zeroes.

(b) Now suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set, where $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$ for each i. Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a set of three linearly independent vectors in $\mathbf{R}^3$, it must form a basis (see Theorem 9 on p. 207), and therefore every vector in $\mathbf{R}^3$ can be written as a linear combination of these.

Now let $\mathbf{v}$ be any vector in $\mathbf{R}^3$. I want to show that $p(A)\mathbf{v} = \mathbf{0}$; I can then apply part (a) to conclude that $p(A)$ is the zero matrix. Write $\mathbf{v}$ as a linear combination of the vectors $\mathbf{u}_i$:

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3.$$

Now compute $p(A)\mathbf{v}$:

$$\begin{aligned} p(A)\mathbf{v} &= p(A)\left(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3\right) \\ &= p(A)a_1\mathbf{u}_1 + p(A)a_2\mathbf{u}_2 + p(A)a_3\mathbf{u}_3 \\ &= a_1 p(A)\mathbf{u}_1 + a_2 p(A)\mathbf{u}_2 + a_3 p(A)\mathbf{u}_3 \end{aligned}$$

Now apply the previous problem:

$$= a_1 p(\lambda_1)\mathbf{u}_1 + a_2 p(\lambda_2)\mathbf{u}_2 + a_3 p(\lambda_3)\mathbf{u}_3.$$

The numbers λ_1, λ_2, and λ_3 are roots of $p(t)$, so $p(\lambda_i) = 0$ for all i. Therefore the above sum is equal to the zero vector: $p(A)\mathbf{v} = \mathbf{0}$. This is exactly what I wanted to show. I can conclude that $p(A)$ is the zero matrix.