**Solutions, homework assignment 6**

**Section 4.2: 23**: The matrices in question are

$$A_2 = \begin{bmatrix} d & 1 \\ 1 & d \end{bmatrix}, \; A_3 = \begin{bmatrix} d & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{bmatrix}, \quad \text{and} \quad A_4 = \begin{bmatrix} d & 1 & 1 & 1 \\ 1 & d & 1 & 1 \\ 1 & 1 & d & 1 \\ 1 & 1 & 1 & d \end{bmatrix}.$$

The determinant of $A_2$ is $d^2 - 1 = (d-1)(d+1)$, which fits the formula.

To compute the determinant of $A_3$, expand along the first row:

$$\begin{aligned}
\det(A_3) &= d \cdot \det(A_2) - \begin{vmatrix} 1 & 1 \\ 1 & d \end{vmatrix} + \begin{vmatrix} 1 & d \\ 1 & 1 \end{vmatrix} \\
&= d(d-1)(d+1) - 2(d-1) \\
&= (d-1)((d^2+d)-2) \qquad \text{(factor out } d-1\text{)} \\
&= (d-1)(d-1)(d+2) \\
&= (d-1)^2(d+2).
\end{aligned}$$

This fits the formula.

To compute the determinant of $A_4$, expand along the first row:

$$\begin{aligned}
\det(A_4) &= d \cdot \det(A_3) - \begin{vmatrix} 1 & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{vmatrix} + \begin{vmatrix} 1 & d & 1 \\ 1 & 1 & 1 \\ 1 & 1 & d \end{vmatrix} - \begin{vmatrix} 1 & d & 1 \\ 1 & 1 & d \\ 1 & 1 & 1 \end{vmatrix} \\
&= d(d-1)^2(d+2) - 3\begin{vmatrix} 1 & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{vmatrix} \qquad \text{(switch rows around in the last two matrices)} \\
&= (d-1)^2(d^2+2d) - 3(d-1)^2 \qquad \text{(expand along first row)} \\
&= (d-1)^2(d^2+2d-3) \\
&= (d-1)^2(d-1)(d+3) \\
&= (d-1)^2(d+3).
\end{aligned}$$

This is what it's supposed to be.

Here's another solution based on one that several people turned in, and I'll just do the $4 \times 4$ case (although the method works for matrices of any size): first add rows 1, 2, and 3 to row 4. This doesn't change the determinant, but it turns the matrix into

$$\begin{bmatrix} d & 1 & 1 & 1 \\ 1 & d & 1 & 1 \\ 1 & 1 & d & 1 \\ d+3 & d+3 & d+3 & d+3 \end{bmatrix}.$$

By properties of determinants, I get

$$\det \begin{bmatrix} d & 1 & 1 & 1 \\ 1 & d & 1 & 1 \\ 1 & 1 & d & 1 \\ d+3 & d+3 & d+3 & d+3 \end{bmatrix} = (d+3)\det \begin{bmatrix} d & 1 & 1 & 1 \\ 1 & d & 1 & 1 \\ 1 & 1 & d & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

In this last matrix, subtract the last row from each of the others:

$$= (d+3)\det \begin{bmatrix} d-1 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 0 \\ 0 & 0 & d-1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is lower triangular, so it's determinant is easy to compute:

$$= (d+3)(d-1)^3.$$

**Section 4.3: 28**: Suppose that $A$ is skew symmetric: suppose that $A^T = -A$. Therefore $\det(A^T) = \det(-A)$. By Theorem 5, the left side of this equals $\det(A)$. Since $-A$ is formed from $A$ by multiplying each of the $n$ rows by $-1$, I can apply Theorem 7 $n$ times to get $\det(-A) = (-1)^n \det(A)$. Therefore

$$\det(A) = (-1)^n \det(A).$$

If $n$ is odd, then $(-1)^n$ equals $-1$, so this equation becomes

$$\det(A) = -\det(A),$$

which means that $\det(A) = 0$. By Theorem 3 (p. 287), this means that $A$ is singular.