Solutions, homework assignment 4

Section 3.5, 36: The rank of A is defined to be the dimension of the range. Since the range is equal to the column space of A, the rank is therefore equal to the dimension of the column space. Since A by assumption has n columns, its column space is spanned by n vectors, and so its dimension is at most n. Thus $\text{rk}(A) \leq n$. By theorem 10 in the book, $\text{rk}(A) = \text{rk}(A^T)$. A^T has m columns, so its rank is at most m, and thus $\text{rk}(A) \leq m$.

Another way to show that $\text{rk}(A) \leq m$: the range of A is a subspace of $\mathbf{R}^m$, and every subspace of $\mathbf{R}^m$ has dimension at most m. Hence $\text{rk}(A) \leq m$.

Another way to show that $\text{rk}(A) \leq n$: the rank-nullity theorem says that

$$\text{rk}(A) + \text{nullity}(A) = n.$$

The nullity of A cannot be negative, so $\text{rk}(A) \leq n$.

Section 3.5, 38: The the rank-nullity formula (the remark at the bottom of page 209) says that if A is $m \times n$, then

$$n = \text{rk}(A) + \text{nullity}(A).$$

So if A is 3×4 with nullity 1, then its rank is 3. Thus its range has dimension 3, and since the range is a subspace of $\mathbf{R}^3$, the range must be all of $\mathbf{R}^3$. Now, the range consists of all vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent (I've discussed this in class; also see the sentence after the definition of range on page 181 in the book). Therefore, for every vector $\mathbf{b}$ in $\mathbf{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ is consistent.